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HISTORY

*Bruins, E. M. **Fontes matheseos. Hoofdpunten van het prae-griekse en griekse wiskundig denken.** [**Fontes matheseos. Main features of pre-Greek and Greek mathematical thought.**] E. J. Brill, Leiden, 1953. xii + 168 pp. (6 plates). 7.15 guilders.

This handy book is primarily written for the use of pupils in Dutch gymnasia (Latin high schools). It contains a number of original Greek texts with Dutch introduction (but no translation) selected so as to elucidate the treatment of simple problems in Greek mathematics. The texts are taken from Heron's "Metrika" (mainly a number of prescriptions on the determination of areas, volumes, square and cubic roots), Plato's "Menon" (the diagonal of the square), Archimedes (computation of π, the summation of the first n squares, the area of the parabolic segment, the Arenaria and the hydrostatic law), and Ptolemy (determination of the table of chords). The book opens with a concise 23 page introduction into Egyptian and Babylonian mathematics, with an explanation of a hieratic text on fractions, which is reproduced. Also reproduced are some Babylonian texts, as well as a page of the Constantinople Heron codex. There are 71 figures.

Euclid's "Elements" are excluded from his collection, mainly because the author feels that it is undesirable to lift a fragment from an axiomatic structure; moreover, knowledge of the "Elements" is not indispensable for the correct understanding of many philosophical writings.

D. J. Struik (Cambridge, Mass.).

van der Waerden, Bartel L. Die Bewegung der Sonne nach griechischen und indischen Tafeln. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1952, 219–232 (1953).

The author shows that Greek tables for the travel of the sun through the single signs of the zodiac as well as tables used in Tamil astronomy for the solar motion were based on trigonometric methods (in contrast to the Babylonian "linear" methods). For the Greek tables one obtains agreement with the excenter motion as used by Hipparchus and Ptolemy. For the Tamil tables a new type of model is reconstructed: a concenter with equant (of longitude 78° and distance 0; 2, 15 if $r=1$). *O. Neugebauer.*

*Schmidt, Olaf. **Some critical remarks about Autolycus' On risings and settings.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 202–209. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

A careful discussion of the contents of Autolycus' "On risings and settings" leads to the remarkable result that the traditional arrangement in two "books" is actually a repetition of two versions of the same work. This contamination must have happened before the 10th century since all available manuscripts show the same division [cf. the edition by Mogenet, Louvain, 1950, p. 156—Mogenet did not know of Schmidt's discovery]. *O. Neugebauer.*

*Apostle, Hippocrates George. **Aristotle's philosophy of mathematics.** The University of Chicago Press, Chicago, Ill., 1952. x+228 pp. $6.00.

Combining relevant fragments from Aristotle's extant works, the author tries to reconstruct his philosophy of mathematics. Leaving detailed criticism of the author's treatment of the texts to professional philologists, it can be said that indeed a readable, vivid, complete, and reliable account of the Stagirite's views on the subject is given. Philosophers and mathematicians interested in Aristotle's views on the method and subject-matter of mathematics, on the relation of arithmetic and geometry and of pure and applied mathematics, on the infinite, on continuity, and so on, will find in this book a valuable tool. Critical comments: (1) More attention has been given to the subject than is suggested by the Preface; the reviewer's "Philosophy of mathematics from Parmenides to Bolzano" [Standaard, Antwerpen, 1944 (Dutch)] lists some 25 relevant publications. (2) Neglect of previous work explains, perhaps, such an anachronism as the introduction of a definition of ratio. (3) Definitions introduced by the author are not marked; there is no index on passages and no bibliography. (4) In the absence of a commentary written from the contemporary point of view, non-specialists will have many difficulties in grasping the purport and even the relevance of Aristotle's discussions. For instance, in connection with the material contained in Chapter V, "Criticism of various views on mathematics", the simple remark that here the distinction between pure and applied mathematics is at stake, would have been helpful. (5) The author rightly states that Aristotle's account of other philosophies of mathematics is fairly accurate; but it is equally true that it is often difficult to understand it without reference to other data. The author's method of strictly adhering to an English-Greek dictionary of important philosophical terms (included in the book) is most laudable. *E. W. Beth* (Amsterdam).

Taton, René. Les mathématiques selon l'Encyclopédie. Rev. Hist. Sci. Appl. 4, 255–266 (1951).

Costabel, Pierre. La mécanique dans l'Encyclopédie. Rev. Hist. Sci. Appl. 4, 267–293 (1951).

Larsen, L. Melchior. Sketch of the history of the art of computing in Denmark. Mat. Tidsskr. A. 1952, 1–21 (1952). (Danish)

Yanovskaya, S. A. Two documents on the history of Moscow University. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 1952, no. 8, 41–56 (1952). (Russian)

The documents consist of an essay by A. S. Eršov, "On instruction in technical mechanics and descriptive geometry" and comments on this essay by Čebyšev (unsigned but recognizable by the handwriting).

Brun, Viggo. Niels Henrik Abel. Norske Vid. Selsk. Forh., Trondheim **25** (1952), 25*–43* (1953).

von Krbek, F. Geschichte des Prinzips von d'Alembert. Wissensch. Z. Univ. Greifswald. Math.-Nat. Reihe **2**, 15–22 (1953).

***Apollonius of Perga. On conic sections.** Great Books of the Western World, no. 11, pp. 593–804. Encyclopaedia Britannica, Inc., Chicago, London, Toronto, 1952.
This edition is from the translation of R. C. Taliaferro.

Procissi, Angiolo. La traduzione italiana delle Opere di Archimede nelle carte inedite di Vincenzo Viviani (1622–1703). Boll. Un. Mat. Ital. (3) **8**, 74–82 (1953).

de la Cinta Badillo, M.ª. George Berkeley. Gaceta Mat. (1) **4**, 233–235 (1 plate) (1952). (Spanish)

Thorndike, Lynn. Giovanni Bianchini in Italian manuscripts. Scripta Math. **19**, 5–17 (1953).

Emanuele, Maria Antonietta. La trigonometria generale secondo Giovanni Bolyai. Matematiche, Catania **7**, 18–20 (1952).

Saltykow, N. La vie et l'oeuvre de Elie Cartan. Bull. Soc. Math. Phys. Serbie **4**, no. 3–4, 59–64 (1 plate) (1952). (Serbo-Croatian. French summary)

Godeaux, Lucien. Guido Castelnuovo, Frederigo Enriques et la géométrie algébrique. Rev. Gén. Sci. Pures Appl. **60**, 8–14 (1953).

Tenca, Luigi. La corrispondenza epistolare fra Tomaso Ceva e Guido Grandi. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 519–537 (1951).

Brunet, Pierre. La vie et l'oeuvre de Clairaut. Rev. Hist. Sci. Appl. **4**, 13–40, 109–153 (1951); **5**, 334–349 (1952); **6**, 1–17 (1953).

***Copernicus, Nicolaus. On the revolutions of the heavenly spheres.** Great Books of the Western World, no. 16, pp. 497–838. Encyclopaedia Britannica, Inc., Chicago, London, Toronto, 1952.
The translation is by C. G. Wallis.

Savin, G. N. Obituary: Aleksadr Nikolaevič Dinnik (1876–1950). Ukrain. Mat. Žurnal **3**, 123–127 (1951). (Russian)

Obituary: Kerim Erim. Rev. Fac. Sci. Univ. Istanbul (A) **18**, i–iv (1 plate) (1953). (Turkish and French)

Hackett, F. E. Fitzgerald as revealed by his letters to Heaviside. Sci. Proc. Roy. Dublin Soc. (N.S.) **26**, 3–7 (1952).

Sunouchi, G. Supplementary list of mathematical publications by Prof. M. Fujiwara. Tôhoku Math. J. (2) **4**, 316–317 (1952).
This completes a list of Fujiwara's papers published in the same J. **49**, 133–138 (1943); these Rev. **7**, 355.

Bridgman, P. W., Quine, W. V., Van Vleck, J. H., and Widder, D. V. Obituary: Edward Vermilye Huntington. 1874–1952. Bull. Amer. Math. Soc. **59**, 399 (1953).

***Kepler, Johannes. Epitome of Copernican astronomy. Books IV and V.** Great Books of the Western World, no. 16, pp. 839–1004. Encyclopaedia Britannica, Inc., Chicago, London, Toronto, 1952.
From the translation by C. G. Wallis.

***Kepler, Johannes. The harmonies of the world.** Great Books of the Western World, no. 16, pp. 1005–1085. Encyclopaedia Britannica, Inc., Chicago, London, Toronto, 1952.
The translation is by C. G. Wallis.

Sasaki, Shigeo. Obituary note: Tadahiko Kubota (1885–1952). Tôhoku Math. J. (2) **4**, 318–319 (1952).

Sasaki, Shigeo. Supplementary lists of mathematical publications of Prof. T. Kubota. Tôhoku Math. J. (2) **4**, 321–322 (1952).
This supplements an earlier list published in the same J. (2) **1**, 3–13 (1949); these Rev. **11**, 573.

Severi, F. Leonardo e la matematica. Scientia **88**, 41–44 (1953).

Natucci, A. Leonardo geometra. Giorn. Mat. Battaglini (5) **1(81)**, 89–103 (1952).

***Norden, A. P. 125 years of non-Euclidean geometry.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 13–22. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

***Laptev, B. L. Life and activity of N. I. Lobačevskiĭ.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 23–33. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

***Rybkin, G. F. On the Weltanschauung of N. I. Lobačevskiĭ.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 43–60. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

***Bronštein, I. N. Disclosure of the legacy of N. I. Lobačevskiĭ and of materials for his biography.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 61–74. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

***Morozov, V. V. On the algebraic manuscripts of N. I. Lobačevskiĭ.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 75–78. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

***Gagaev, B. M. Generalization of the Fourier integral by N. I. Lobačevskiĭ.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 79–86. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

*Dubyago, A. D. **N. I. Lobačevskiĭ's trip to Penza to observe the solar eclipse of 1842.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 87–98. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

*Laptev, B. L. **The theory of parallel lines in early works of N. I. Lobačevskiĭ.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 99–116. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

*Norden, A. P. **On the exposition of the fundamental theorems of Lobačevskiĭ's geometry.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 117–128. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

Prudnikov, V. E. The first Russian arithmetician and geometer. (For the 250th anniversary of the publication of L. F. Magnickiĭ's "Arithmetic".) Mat. v. Skole **1953**, no. 2, 12–15 (1953). (Russian)

Mineo, Corradino. In memoria di Gaspare Mignosi (5 Gennaio 1875–11 Giugno 1951). Matematiche, Catania **7**, iii–xii (1 plate) (1952).

A list of Mignosi's published mathematical work is included.

Vandiver, H. S. Les travaux mathématiques de Dmitry Mirimanoff. Enseignement Math. **39** (1942–1950), 169–179 (1953).

Carlson, A. J. Obituary: Forest Ray Moulton: 1872–1952. Science **117**, 545–546 (1953).

*Newton, I. S. **Philosophiae naturalis principia mathematica.** William Dawson & Sons, Ltd., London, undated. viii + 510 + i pp. 84 shillings.

Facsimile reproduction of the edition of 1686.

Mihail Vasil'evič Ostrogradskiĭ. (On the 150th anniversary of his birth.) Ukrain. Mat. Žurnal **3**, 235–239 (1 plate) (1951). (Russian)

Štokalo, I. Z. Works of M. V. Ostrogradskiĭ in mathematical physics. Ukrain Mat. Žurnal **4**, 3–24 (1952). (Russian)

Svešnikov, A. G. On a work of M. V. Ostrogradskiĭ. Uspehi Matem. Nauk (N.S.) **8**, no. 1(53), 101–102 (1953). (Russian)

The author states that a work of Ostrogradskiĭ dating from 1828 [Note sur la théorie de la chaleur, Mém. Acad. Sci. St. Pétersbourg. Sci. Math. Phys. Nat. (6) **1**, 129–138 (1831)] demonstrates his prior discovery of several mathematical theorems and developments usually credited to other mathematicians. There follows (pp. 102–103) an extract relating to this paper taken from a speech of Steklov given at a celebration of the 100th anniversary of Ostrogradskiĭ's birth and a Russian translation of the paper in question (pp. 103–110).

*Ptolemy. **The almagest.** Great Books of the Western World, no. 16, pp. vii–xiv + 1–478. Encyclopaedia Britannica, Inc., Chicago, London, Toronto, 1952.

The translation is by R. C. Taliaferro.

Conte, Luigi. Le quattro regole di William Purser. Period. Mat. (4) **31**, 1–6 (1953).

Archibald, R. C. The Canon Doctrinae Triangvlorvm (1551) of Rheticus (1514–1576). Math. Tables and Other Aids to Computation **7**, 131 (1953).

Hofmann, J.-E. A propos d'un problème de Roberval. Rev. Hist. Sci. Appl. **5**, 312–333 (1952).

*Schläfli, Ludwig. **Gesammelte mathematische Abhandlungen. Band II.** Verlag Birkhäuser, Basel, 1953. 381 pp. 56.15 Swiss francs.

This second volume begins with a long article on the resultant of a system of algebraic equations, with comments by Burckhardt and van der Waerden. Next come two papers on the coefficients of powers of x in the expansion of

$$(1+x)(1+2x)\cdots\{1+(n-1)x\},$$

and a short one on confocal quadrics in n dimensions. The note "Über eine Funktion von drei Winkeln . . ." is of special interest because it shows that Schläfli knew how to find the volume of a tetrahedron not only in spherical space but also in hyperbolic space, although at that time (1852) he was almost certainly unaware of Lobačevskiĭ's work. This volume also contains the French and English versions of his "Theorie der vielfachen Kontinuität" [same Abhandlungen, vol. 1, 1950, pp. 167–392; these Rev. 11, 611], separated by two other papers, one of which is modestly entitled "An attempt to determine the twenty-seven lines upon a surface of the third order, and to divide such surfaces into species in reference to the reality of the lines upon the surface." The existence of 27 such lines had already been discovered by Cayley and Salmon, but this paper of 1856 gives the first complete description of the configuration (in the celebrated notation a_i, b_j, c_{ij}). The author's versatility is further exemplified by a note on the acceleration of a planet in an elliptic orbit, and another giving a neat proof of von Staudt's theorem about the fractional parts of Bernoulli numbers. Finally, there are the two papers of 1865 in which Schläfli announced his discovery of the decomposition of the general n-dimensional orthogonal transformation into $[\frac{1}{2}n]$ rotations.

As in the case of vol. 1, the printing has been very well done, with only a few misprints. But the collection might have been even more useful if the various papers had been arranged in a more systematic order: either chronological or according to subjects. *H. S. M. Coxeter.*

Signorini, Antonio. Elenco cronologico delle pubblicazioni. Rivista Mat. Univ. Parma **3**, 301–306 (1952).

Dutka, Jacques. Spinoza and the theory of probability. Scripta Math. **19**, 24–33 (1953).

Sestini, G. Obituary: Pietro Teofilato (28 agosto 1879–31 agosto 1952). Rivista Mat. Univ. Parma **3**, 291–296 (1952).

Procissi, Angiolo. Su l'inviluppo delle parabole in Torricelli e sulla nozione di inviluppo di una famiglia di curve piane. Period. Mat. (4) **31**, 34–43 (1953).

FOUNDATIONS

Ackermann, Wilhelm. Widerspruchsfreier Aufbau einer typenfreien Logik. II. Math. Z. **57**, 155–166 (1953).

In part I [Math. Z. **55**, 364–384 (1952); these Rev. **14**, 344] a type-free system was set up which can be proved consistent and in which certain instances of the law of the excluded middle are provable. For a substantial part of mathematics a weaker system is sufficient, namely, the part which does not contain the connective $\rightarrow$. Using this system, the present paper treats number theory, the theory of rational and real numbers, and part of analysis.

I. Novak Gái (Ithaca, N. Y.).

Sobociński, Bolesław. Axiomatization of a partial system of three-value calculus of propositions. J. Computing Systems **1**, 23–55 (1952).

The author studies a 3-valued logic distinct from that of Łukasiewicz, and gives an axiomatization of it. The primitive terms are implication and negation. The system is determined by a matrix with two designated elements, whereas the Łukasiewicz matrix has only one designated element. The corresponding formal system has substitution and modus ponens as its rules of procedure, and the following five axioms: (1) $(p \supset q) \supset ((q \supset r) \supset (p \supset r))$; (2) $p \supset ((p \supset q) \supset q)$; (3) $(p \supset (p \supset q)) \supset (p \supset q)$; (4) $p \supset (q \supset (q' \supset p))$; and (5) $(p' \supset q') \supset (q \supset p)$. It is proved that a formula involving implication and negation is valid in the 3-valued system if and only if it is a consequence of this axiom system. It is also shown that the axioms are independent.

After 182 consequences of the axiom system are derived in succession, it is shown that all valid formulas of the 3-valued system are consequences of them. The chief merit claimed for the system is that it avoids all except four of the recognized paradoxical formulas of the two-valued system of material implication, though making no use of modalities. It contains also a large proportion of the common non-paradoxical two-valued formulas. It is shown that the degree of extension of the system is three. Operations of conjunction, disjunction, and equivalence may be defined. The author also compares and contrasts his system with that of Łukasiewicz. *O. Frink* (State College, Pa.).

Rose, Alan. Le degré de saturation du calcul propositionnel implicatif à trois valeurs de Sobociński. C. R. Acad. Sci. Paris **235**, 1000–1002 (1952).

Sobociński has studied a 3-valued logic distinct from that of Łukasiewicz and based on a matrix for the operations of implication and negation [see the preceding review]. He showed that this system had its degree of completeness (this is a notion due to Tarski) equal to 3, but did not answer the corresponding question for the implicative system obtained by ignoring the negation operation. The author answers this question by proving that any weakly complete formalization of the Sobociński implicative system having substitution and modus ponens for its rules of procedure, has its degree of completeness also equal to 3. *O. Frink.*

Rose, Alan. The degree of completeness of the $\aleph_0$-valued Łukasiewicz propositional calculus. J. London Math. Soc. **28**, 176–184 (1953).

Łukasiewicz has developed a countably-infinitely-valued propositional calculus with one designated element and having the rational numbers from 0 to 1 for its truth values. No complete formalization of this system is known. It has been conjectured by Łukasiewicz that a complete formalization is given by a certain set of five axioms, with substitution and modus ponens as the rules of procedure.

Tarski [Monatsh. Math. Phys. **37**, 361–404 (1930)] has defined the notions of cardinal and ordinal degree of completeness of a formal system. The author shows that every weakly complete formalization of the countably-infinitely-valued Łukasiewicz propositional calculus having substitution and modus ponens as its only rules of procedure has its cardinal degree of completeness equal to aleph-null, and its ordinal degree of completeness equal to ω. *O. Frink.*

Götlind, Erik. Some Sheffer functions in n-valued logic. Portugaliae Math. **11**, 141–149 (1952).

By a Sheffer function the author means a function of two variables $F(p, q)$ in terms of which all functions of the n-valued logical calculus may be defined. He shows that a certain set of conditions are sufficient for $F(p, q)$ to be a Sheffer function. These conditions are more general than those given by N. M. Martin [Methodos **3**, 240–242 (1951)]. *O. Frink* (State College, Pa.).

Carruccio, Ettore. Sulle dimostrazioni di coerenza dei sistemi ipotetico-deduttivi. Univ. e Politecnico Torino. Rend. Sem. Mat. **10**, 97–110 (1951).

This contains some general observations on the difficulty of proving the consistency of any formal logical system S adequate for analysis. On the one hand Gödel's theorem shows the impossibility of proving consistency by means which may be formalized within S. On the other hand attempts at proving the consistency of S by making it dependent on the consistency of another system Σ also involve difficulties, it is claimed. Included under this heading are programs involving consistency proofs to be carried out in some meta-language. *O. Frink* (State College, Pa.).

Greniewski, Henryk. Groups and fields definable in the propositional calculus. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **43** (1950), 53–68 (1952).

This deals with the question of defining finite Abelian groups and finite (Galois) fields in terms of the notation of the calculus of elementary propositions, excluding quantifiers and the logical constants 0 and 1. The group and field concepts are first defined in the first order functional calculus using quantifiers, and "analogs" of the formulas found there are then sought in the calculus of elementary propositions. This is an interesting program; the reviewer was unable to determine whether the author was completely successful in carrying it out. *O. Frink* (State College, Pa.).

Shepherdson, J. C. Inner models for set theory. II. J. Symbolic Logic **17**, 225–237 (1952).

The object of this series of three papers is to show that for a fairly large class of inner models, that is, models whose domain of individuals is a certain subset of the original individual domain, no further essentially new results can be proved. This, the second paper of the series, treats this problem for super-complete models. If, as in part I [same J. **16**, 161–190 (1951); these Rev. **13**, 522], the notation and axioms A, B, C, D, E are those used in Gödel, K: The consistency of the continuum hypothesis [Princeton, 1940; these Rev. **2**, 66], then a class V_m is said to determine a super-complete model of a set-theoretical universe

satisfying axioms A, B, C if the model whose basic notions are defined by $\mathfrak{Cls}_m(X).\equiv.X\subseteq V_m$, $\mathfrak{M}_m(X).\equiv.X\,\varepsilon\,V_m$, $X\,\varepsilon\,Y:\equiv\mathfrak{M}_m(X)\cdot\mathfrak{Cls}_m(Y)\cdot X\,\varepsilon\,Y$ satisfies axioms A, B, C. Let the function G over the ordinals be defined inductively as follows: $G'0=0$, for $\alpha>0$, $x\,\varepsilon\,G'\alpha.\equiv.(\exists\gamma)(\gamma<\alpha.x\subseteq G'\gamma)$. Then one can prove: In a universe satisfying axioms A, B, C, D, E the set $G'\rho$, where ρ is an inaccessible ordinal (in the narrow sense) determines a super-complete model for which $On_m=\rho$; furthermore, the only non-trivial super-complete models (i.e., distinct from that formed by the universe itself) are those determined by such sets. Moreover, for super-complete models, 'V=L' is absolute. Lastly there are two theorems concerning the consistency of the non-existence of inaccessible ordinals, reproving theorems of Firestone and Rosser [J. Symbolic Logic 14, 79 (1949)]: 1) If the system of axioms A, B, C is consistent, then so is the system A, B, C, D, E, 'there are no inaccessible ordinals (in the narrow sense)'. 2) If the system of axioms A, B, C is consistent, then so is the system A, B, C, D, E, 'V=L', 'there are no inaccessible ordinals (in the wide sense)'.

I. Novak Gál (Ithaca, N. Y.).

Lorenzen, Paul. Eine Bemerkung über die Abzählbarkeitsvoraussetzung in der Algebra. Math. Z. 57, 241–243 (1953).

Let G be a primary abelian group whose elements are of finite order and height. Prüfer [Math. Z. 17, 35–61 (1923)] proved that if G is countable, then it can be represented as the direct sum of cyclic subgroups. Counter-examples constructed by Prüfer and Ulm [Ulm, ibid. 40, 205–207 (1935)] show that the theorem is not generally true if G is non-countable. In the present paper, the author reviews the problem from the point of view of his own constructivist foundations for mathematics [ibid. 53, 162–202 (1950); 54, 1–24 (1951); these Rev. 12, 469; 13, 310]. According to that theory the various notions of mathematics are divided into layers or strata, depending on their mode of construction, and a set may be non-countable in one layer but countable in terms of a higher layer. It follows that the decomposition of G is always possible in a suitable layer, and it is shown that Ulm's example can be re-interpreted accordingly.

The author adds some general observations which imply that the distinction between transfinite sets of different cardinal numbers is illusory in algebra (though not in analysis). He maintains that every algebraic theorem that is true on the assumption of countability must be true generally.

If by "a theorem in algebra" is meant a theorem which can be formulated in the restricted predicate calculus, then the author's assertion will be accepted even by a non-constructivist reader, for it is a direct consequence of the theorem of Skolem-Löwenheim. However, the conclusion of Prüfer's theorem cannot be so formulated, and from the classical point of view one clearly cannot accept the present author's conclusions as well as Ulm's reasoning, since the two contradict each other. Without committing oneself to one school of thought or the other one may say that the present paper has the merit of showing that contemporary arguments concerning the relativity of transfinite concepts may be of importance even in connection with (what appears to be) a perfectly concrete problem in algebra.

A. Robinson (Toronto, Ont.).

Grünbaum, Adolf. A consistent conception of the extended linear continuum as an aggregate of unextended elements. Philos. Sci. 19, 288–306 (1952).

By philosophers it is still often believed that Zeno's paradoxes have shown that no conception of an extended continuum as an aggregate of unextended points can be consistent. On the basis of contemporary mathematical theories (topology, theory of cardinal numbers, measure theory) presupposing Cantor's theory of sets, the author shows that this belief is untenable. His exposition provides a striking illustration of the importance of looking at older doctrines in the light of modern science. It is not true that (as the author states on p. 303) the intuitionists "cannot avail themselves of a super-denumerable set"; Brouwer's theory of spreads, based upon the notion of a free choice sequence, is intended as a substitute for Cantor's theory of sets and in particular it provides an intuitionistic construction of the continuum of real numbers. *E. W. Beth.*

Osborn, Roger. A consideration of the null class. Math. Mag. 26, 175–182 (1953).

***Yin, Fu-Sheng. On categorical syllogism.** Essays and papers in memory of late president Fu SSu-Nien, pp. 109–122. The National Taiwan University, Taipei, 1952. (Chinese)

The reviewer has not been able to find anything new or interesting in this article. Nor is it easy to summarize the content, since it is not clear to the reviewer what are the points which the author wishes to make. According to the author, it is "on" categorical syllogism in three senses: as description, as critique, and as modification and correction.

Hao Wang (Philadelphia, Pa.).

Reidemeister, Kurt. Zur Logik der Lehre vom Raum. Dialectica 6, 327–342 (1952).

A philosophical argument is given to show that while there cannot be complete equivalence between Euclidean geometry in the exact sense, actual physical space, and our intuition of physical space, nevertheless there is a certain rigorous sub-theory of all three, related to combinatorial topology, which is at once exact, intuitive, and physically real. To illustrate this point, he gives a proof of the Euler polyhedral theorem for finite triangulations of a square. The proof makes use of properties of the integers only, not of the real number system. *O. Frink.*

Lorent, Henri. Réalités et symbolismes en mathématiques. Enseignement Math. 39 (1942–1950), 194–210 (1953).

***Coulson, C. A. The spirit of applied mathematics.** Oxford, at the Clarendon Press, 1953. 23 pp. $.60.

Inaugural lecture delivered before the University of Oxford on 28 October 1952.

ALGEBRA

Harary, Frank, and Uhlenbeck, George E. On the number of Husimi trees. I. Proc. Nat. Acad. Sci. U. S. A. **39**, 315–322 (1953).

In connection with the virial expansion of the equation of state for non-ideal gases, the combinatorial problem of counting certain types of linear graphs arises. The paper outlines the solution of this problem for the so-called Husimi trees [K. Husimi, J. Chem. Phys. **18**, 682–684 (1950); these Rev. **12**, 467], defined as connected graphs in which no line lies on more than one cycle. A pure Husimi tree is one which consists of cycles having all the same number m of lines. The well-known Cayley trees correspond to $m=1$. Following the methods developed for Cayley trees, the authors derive a functional equation for the generating functions $T^{(m)}(x)$ of the numbers of rooted pure Husimi trees (i.e., trees with one preferred point, all other points being equivalent) and express in terms of $T^{(m)}(x)$ the generating function for free pure Husimi trees (with all points equivalent) and free general Husimi trees. *L. Van Hove* (Princeton, N. J.).

Nair, K. R. A note on rectangular lattices. Biometrics **9**, 101–106 (1953).

The author shows that the n-fold rectangular lattice for $p(p-1)$ varieties in blocks of $(p-1)$ plots, where every variety is repeated n times, is a partially balanced incomplete block design (p.b.i.b.d.) with 4, 3 and 2 associate classes if $n=2$ $(p\geqq 4)$, $n=p-1$, and $n=p$, respectively. It is conjectured that the designs are not p.b.i.b.d. for any other value of n. If $n=p$, the dual design is a p.b.i.b.d. with two associate classes. The author gives its parameters and its efficiency factor. Several examples are given.

H. B. Mann (Columbus, Ohio).

Rédei, L. Kurzer Beweis der Waringschen Formel. Acta Math. Acad. Sci. Hungar. **3**, 151–153 (1952). (Russian summary)

A simple proof, by induction, of the formula, due to Waring, that expresses $x_1^h+x_2^h+\cdots+x_n^h$ in terms of the elementary symmetric functions of $x_1, x_2, \cdots, x_n$.

H. W. Brinkmann (Swarthmore, Pa.).

Pipping, Nils. Verallgemeinerungen des Euklidischen Algorithmus. Acta Acad. Aboensis **18**, no. 5, 17 pp. (1952).

The author considers Jacobi's generalization of the continued fraction algorithm and three variants of it due to V. Brun, L. Törnqvist, and the author. As applied to the expansion of the positive root w of a cubic equation with rational coefficients, the question under consideration is whether a given algorithm will disclose the fact that w is a cubic irrationality by terminating or becoming periodic. Examples are exhibited showing widely varying behavior of the several algorithms. A proof is given of the fact that the algorithm of Törnqvist terminates or becomes periodic if and only if w is an algebraic number of degree $\leqq 3$.

D. H. Lehmer (Los Angeles, Calif.).

Brauer, Alfred. Limits for the characteristic roots of a matrix. V. Duke Math. J. **19**, 553–562 (1952).

The four earlier parts in this sequence were published in the same J. **13**, 387–395 (1946); **14**, 21–26 (1947); **15**, 871–877 (1948); **19**, 75–91 (1952) [these Rev. **8**, 192, 559; **10**, 231; **13**, 813] and will be referred to by roman numerals.

In II it was proved that each characteristic root of a square matrix $A=(a_{\kappa\lambda})$ of order n lies in at least one of the ovals of Cassini $|z-a_{\kappa\kappa}|\cdot|z-a_{\lambda\lambda}|\leqq P_\kappa P_\lambda$, where $P_\kappa=\sum_{\mu=1}^{n}|a_{\kappa\mu}|-|a_{\mu\mu}|$ and $\kappa\neq\lambda$. This is now improved by replacing $P_\kappa P_\lambda$ by

$$P_{\kappa\lambda}=|a_{\kappa\lambda}|P_\lambda+|a_{\lambda\kappa}|(P_\kappa-|a_{\lambda\lambda}|)+\sum_\nu'|a_{\kappa\nu}a_{\lambda\nu}| +\sum_{\nu<\mu}'|a_{\kappa\nu}a_{\lambda\mu}+a_{\kappa\mu}a_{\lambda\nu}|,$$

where $\kappa\neq\lambda$ and in the summations the values κ and λ are omitted. Replacing $P_\kappa P_\lambda$ by $P_{\kappa\lambda}$ improves Theorems 12–19 and 22 of II and III. The proof follows by calculating an upper bound for $|\omega-a_{\tau\tau}|\cdot|\omega-a_{ss}|\cdot|x_\tau|\cdot|x_s|$ from $Ax=\omega x$, where $|x_\tau|\geqq|x_s|\geqq\max|x_\nu|$ and $\nu=1,\cdots,n$; $\nu\neq r, s$.

For a real matrix, $P_{\kappa\lambda}$ is improved, for the real roots only, to $P^*_{\kappa\lambda}$ by replacing the sum of the absolute values of $a_{\kappa\nu}a_{\lambda\nu}$ $(\nu\neq\kappa,\lambda)$ by $\max(u_{\kappa\lambda},|v_{\kappa\lambda}|)$, where $u_{\kappa\lambda}$ is the sum of the positive and $v_{\kappa\lambda}$ the sum of the negative terms among these $n-2$. There are further improvements in this case and under specialization to matrices with non-negative elements. It is also shown that if one of n regions H_m (the definition is too complicated to give here) has no point in common with any of the other $n-1$, then it contains exactly one root. Numerical examples provide proofs that claimed improvements in bounds are genuine. *W. Givens.*

Medlin, Gene W. Bounds for the characteristic roots of matrices with real elements. Duke Math. J. **19**, 563–565 (1952).

[Cf. the preceding review.] The $P^*_{\kappa\lambda}$ are shown to give ovals within which not only the real but all the roots of a real matrix must lie. *W. Givens* (Knoxville, Tenn.).

Brauer, Alfred. Matrices with all their characteristic roots in the interior of the unit circle. J. Elisha Mitchell Sci. Soc. **68**, 180–183 (1952).

[Cf. the two preceding reviews.] If A has real elements, $|a_{\kappa\kappa}|<1$ for $\kappa=1, 2, \cdots, n$ and

$$1-|a_{\kappa\kappa}+a_{\lambda\lambda}|+a_{\kappa\kappa}a_{\lambda\lambda}>P_{\kappa\lambda} \quad \text{or} \quad P^*_{\kappa\lambda},$$

then all the roots of A lie in the interior of the unit circle.

W. Givens (Knoxville, Tenn.).

Parker, W. V. Characteristic roots of a set of matrices. Amer. Math. Monthly **60**, 247–250 (1953).

Let the matrix A of order n with complex elements be written $A=H+iK$ for H and K Hermitian with roots $\alpha_1\leqq\alpha_2\leqq\cdots\leqq\alpha_n$ and $\beta_1\leqq\beta_2\leqq\cdots\leqq\beta_n$, respectively. Then if $\lambda+i\mu$ is a root of A, it must satisfy $\alpha_1\leqq\lambda\leqq\alpha_n$ and $\beta_1\leqq\mu\leqq\beta_n$, and so lies in a rectangle said to be associated with A. For $n=2$ it is proved that the roots of A are the end points of a diameter of the rectangular hyperbola through the vertices of the associated rectangle. The field of values is the ellipse inscribed in the rectangle and with the roots as foci. If H has root α of multiplicity r and K root β of multiplicity s, then for arbitrary unitary U and V, $B=UHU^*+iVKV^*$ has root $\alpha+i\beta$ of multiplicity at least $r+s-n$. If $(x-c)(x-a)^{n-1}=0$ and $(x-d)(x-b)^{n-1}=0$ are the characteristic equations of H and K, respectively, B has the root $a+ib$ of multiplicity $n-2$ and the others are the ends of a diameter of the equilateral hyperbola through (a, b), (a, d), (c, b), and (c, d).

W. Givens (Knoxville, Tenn.).

Vajda, S. Theorems relating to quadratic forms and their discriminant matrices. Proc. Edinburgh Math. Soc. (2) 10, 13–15 (1953).

Let A and B be n by m and m by n matrices with real elements and suppose rank $A=m$ ($\leqq n$). For $M=I_n-AB$, $M'M$ will have as its roots 0 of multiplicity m and 1 of multiplicity $n-m$ if and only if $I_m-BA=0$ and trace $(AB)'(AB)=m$. The unique solution for B is $B=(A'A)^{-1}A'$. The problem arose for $m=1$ in statistics. *W. Givens.*

Nagler, H. On a certain matrix product with specified latent roots. Proc. Edinburgh Math. Soc. (2) 10, 21–24 (1953).

The problem of the preceding review is generalized by allowing A to have rank $r\leqq m$ and asking that the roots of $M'M$ be 0 of multiplicity $n-k$ and 1 of multiplicity k. In terms of a canonical form, necessary and sufficient conditions are found for a solution B to exist. Always, $k\geqq n-r$ and for $k=n-r$ and $M'M=I_k\oplus 0_{n-k}$ the value of M is unique. *W. Givens* (Knoxville, Tenn.).

Haynsworth, Emilie V. Bounds for determinants with dominant main diagonal. Duke Math. J. 20, 199–209 (1953).

Let $A=(a_{ij})$ of order n with complex elements have determinant D_n and set

$$d_i^{(k)}=|a_{ii}|-\sum_{j=1}^{k}|a_{ij}|,\quad S_i^{(k)}=|a_{ii}|+\sum_{j=1}^{k}|a_{ij}|,$$

where in the sums the value $j=i$ is omitted and $k=1,\cdots,n$. Then if $d_i^{(n)}>0$,

$$|D_n|\geqq|a_{11}|\prod_{k=2}^{n}d_k^{(k)}+\sum_{i=2}^{n}[(\sum_{j<i}|a_{ij}|)\prod_{k<i}d_k^{(i)}\prod_{k>i}d_k^{(k)}]$$

and

$$|D_n|\leqq|a_{11}|\prod_{k=2}^{n}S_k^{(k)}-\sum_{i=2}^{n}[(\sum_{j<i}|a_{ij}|)\prod_{k<i}d_k^{(i)}\prod_{k>i}S_k^{(k)}].$$

For a real matrix A, if $a_{ij}>0$, $d_i^{(n)}>0$ and $a_i=\min_{j\neq i}a_{ij}$, then

$$D_n\geqq 2(n-1)(n-2)^{n-1}\prod_{i=1}^{n}a_i.$$

If $a_{ij}<0$ $(i\neq j)$ and $a_{ii}-\sum_{j\neq i}|a_{ij}|=d_i>0$ $(i=1,\cdots,n)$, a lower bound for D_n is $d(n+d)^{n-1}\prod a_i$, where $a_i=\min_j|a_{ij}|$ and $d=\min_i(d_i/a_i)$.

The determinant of an A with positive elements will be positive if there exist nonnegative numbers c_i such that $b_{ij}=a_{ij}-c_i$ are all positive,

$$b_{ii}b_{hh}>(\sum_{j\neq i}b_{ij})(\sum_{j\neq h}b_{hj})\quad(i,h=1,2,\cdots,n),$$

and, for $i,h\neq k$, $i\neq h$,

$$(b_{ii}-b_{ik})(b_{hh}-b_{hk})\geqq(\sum_{j\neq i,h}|b_{ij}-b_{ik}|)(\sum_{j\neq h,k}|b_{hj}-b_{hk}|).$$

Dropping the requirements $a_{ij}>0$ and $b_{ij}>0$, but supposing that $b_{ii}-\sum_{j\neq i}|b_{ij}|>r_ib_i^*$ and $2b_i\geqq b_i^*$, where r_i is the number of negative b_{ij} in the ith row, $b_i=\min_j\{b_{ij};b_{ij}>0\}$ and $b_i^*=\max_{j\neq i}\{b_{ij};b_{ij}>0\}$ with $b_i=b_i^*=0$ if there are no positive terms in the ith row, then again $D_n>0$.

Let A have positive elements and λ be a real root. If $2a_i\geqq a_i^*$ for a_i the minimum and a_i^* the maximum element in the ith row, then $\lambda\geqq\min_i[a_{ii}-\sum_{j\neq i}a_{ij}+(n-2)(2a_i+a_i^*)]$. For A real and $a_i^{**}=\max_{j\neq i}a_{ij}$, if $a_{ii}-\sum_{j\neq i}|a_{ij}|>na_i^{**}$, then the determinant of $B=(b_{ij})$, with $b_{ij}=a_{ij}+x$, is a monotonically increasing function of x. As a consequence, the determinant of $C=Ax+B$ is an increasing function of x for real A and B whenever $a_{ii}\geqq\sum_{j\neq i}|a_{ij}|$ and x is such that C too has a positive dominant main diagonal.

W. Givens (Knoxville, Tenn.).

Room, T. G. A synthesis of the Clifford matrices and its generalization. Amer. J. Math. 74, 967–984 (1952).

The paper falls into three parts. In the first a detailed description is given of the construction of $2r+1$ (Clifford) matrices γ_i of order 2^r satisfying $\gamma_i^2=I$, $\gamma_i\gamma_j=-\gamma_j\gamma_i$ $(i\neq j)$. The γ_i given are essentially direct products of two by two matrices but the appropriate notation is not used and there is no reference to the equivalent construction in the literature [cf., for example, R. Brauer and H. Weyl, same J. 57, 425–449 (1935)].

In part II, quadrics associated with Clifford matrices are studied. If x_i are $2r+1$ (complex) parameters, the matrices $\sum x_i\gamma_i$ define a family Γ of collineations in (2^r-1)-dimensional projective space, denoted by $[2^r-1]$. It is known [cf. Brauer and Weyl, loc. cit., p. 435 or the reviewer, Ann. of Math. (2) 38, 355–385 (1937)] that a unique correlation commutes with all these collineations. This correlation is introduced explicitly and its invariance proved with the case $r=4$ being taken "for convenience of exposition." (The reviewer is of the opinion that the fact that the invariant polarity is with respect to a null-polarity when $r=4s+1$ or $4s+2$, as is stated in a footnote on p. 972, implies that the changes required in the text for $r\neq 4$ are substantial.) The transforms of a point P of [15] under the family Γ will obviously lie in an [8] and will lie on a smaller space if and only if P is on the quadric Q determining the invariant polarity. The product of the polarity and the collineations gives an invariant family of quadrics. The intersection $\mathfrak{X}$, of dimension 10, of all these quadrics is a locus invariant and transitive under products of collineations of Γ. The structure of $\mathfrak{X}$ is summarized in the theorem: "There are on $\mathfrak{X}$: (i) ∞^{10} [4]'s κ, two of which in general do not meet but each meets ∞^6 others in lines, (ii) ∞^{12} [3]'s σ, any pair of which either do not meet, or they may have a point or a line common. Through any plane of $\mathfrak{X}$ there pass one κ and one σ, (iii) ∞^7 quadric sixfolds q on which the generators are solids σ and sections of [4]'s κ." A "mutuality" is established between points, primes, [4]'s, and [10]'s of $\mathfrak{X}$ analogous to the Study triality of points, generator [3]'s, and tangent primes on a quadric in [7], and it is asserted that this extends to a general r.

In part three, various generalizations of Clifford sets are constructed. The matrices are of order n^r, have nth power the identity and commute except that certain pairs satisfy $VU=\epsilon UV$ for ϵ a primitive nth root of unity or $VU=U^2V$.

W. Givens (Knoxville, Tenn.).

Abstract Algebra

Jacobson, Nathan. Lectures in abstract algebra. Vol. II. Linear algebra. D. Van Nostrand Co., Inc., Toronto-New York-London, 1953. xii+280 pp. $5.85.

This is the second of the three volumes dealing with the fundamentals of abstract algebra. The following incomplete list indicates the content: (Finite-dimensional) vector spaces over a division ring; Linear transformations, duality between vector spaces and their conjugate spaces; Cyclic

modules, minimum and characteristic polynomials of a matrix, finitely generated modules over a principal ideal domain, elementary divisor theory, endomorphism ring, linear transformations commutative with a given one; Jordan-Hölder and Krull-Schmidt theorems (for vector spaces with a set of linear transformations as operators); Bilinear forms, symmetric and Hermitian scalar products (the latter with respect to an involutorial automorphism of the underlying division ring); Witt's [J. Reine Angew. Math. 176, 31–44 (1936)] theorem (with amplifications by Pall [Bull. Amer. Math. Soc. 51, 889–893 (1945); these Rev. 7, 358], Kaplansky [Anais Acad. Brasil. Ci. 22, 1–17 (1950); these Rev. 12, 238] and Dieudonné [Sur les groupes classiques, Hermann, Paris, 1948; these Rev. 9, 494]; Euclidean and unitary spaces; Products of vector spaces, tensor spaces; Structure of the ring of (all) linear transformations of a vector space; Infinite-dimensional vector spaces, their Noether bases, dimensionality of the conjugate space (treated by means of the lemmas of Erdös-Kaplansky and Mackey), finite topology for linear transformations, two-sided ideals in the ring of linear transformations, dense rings, the Chevalley-Jacobson density theorem (on irreducible rings of linear transformations) (which generalizes Burnside's and Wedderburn's theorems) [Jacobson, Trans. Amer. Math. Soc. 57, 228–245 (1945); these Rev. 6, 200], isomorphism theorems (which generalize Artin's theorem), dual spaces and dense rings containing finite rank transformations [author, Ann. of Math. (2) 48, 8–21 (1947); these Rev. 8, 433]. The book adopts basically the abstract conceptual viewpoint, but often uses calculational and constructive methods. Expositions are elegant, and are full of new ideas and techniques even when classical results are dealt with. There are many exercises of much interest.

T. Nakayama (Nagoya).

Szász, G. Dense and semi-complemented lattices. Nieuw Arch. Wiskunde (3) **1**, 42–44 (1953).

Theorems: A lattice with 0 is dense (i.e., $x \cap y = 0$ has no non-trivial solutions) if and only if 0 is meet-irreducible. A semi-modular lattice of finite length is semi-complemented if and only if it is complemented. A complete infinitely distributive semi-complemented lattice is complemented.

P. M. Whitman (Silver Spring, Md.).

Utumi, Yuzo. On complemented modular lattices meet-homomorphic to a modular lattice. Kōdai Math. Sem. Rep. **1952**, 99–100 (1952).

Let L be a modular lattice; $e \in L$ is called a μ-element if $x \cap y \leqq e$ implies $(e \cup x) \cap y \leqq e$. Theorem: Let L satisfy the ascending chain condition. Suppose there is a meet-homomorphism θ of L onto a complemented modular lattice such that (1) $(a \cup b)\theta = b\theta$ if $a\theta = 0$; then the inverse image of 0 is an ideal generated by some μ-element e, and $a\theta = b\theta$ if and only if (2) $a \cap x \leqq e$ is equivalent to $b \cap x \leqq e$. Conversely, given a μ-element $e \in L$, then (2) defines an equivalence relation in L yielding a complemented modular lattice meet-homomorphic to L, satisfying (1) where $x\theta$ is the class containing x. The maximal elements of such classes are characterized. *P. M. Whitman* (Silver Spring, Md.).

Utumi, Yuzo. On primary elements of a modular lattice. Kōdai Math. Sem. Rep. **1952**, 101–103 (1952).

Let L be a modular lattice with ascending chain condition; and let Θ be a set of congruences on L, closed under intersection. Denote by $\theta(a)$ the class containing a under $\theta \in \Theta$, and by a_θ the greatest element of $\theta(a)$. Definitions: $q \in L$ is primary with respect to Θ if for every $\theta \in \Theta$, $q = q_\theta$ or $q = I$ (θ); $\theta \in \Theta$ is the radical with respect to Θ of $a \in L$ if θ is the smallest $\phi \in \Theta$ with $a = I$ (ϕ). The author studies representations of elements of L as intersections of primary elements. Sample theorem: Consider a finite irredundant such representation in which no two of the primary elements have the same radical; then in any other representation satisfying the same conditions, the radicals are pairwise identical with those of the first representation. As examples, the methods are applied to the lattice of ideals of a non-commutative ring and to a distributive lattice.

P. M. Whitman (Silver Spring, Md.).

Matsushita, Shin-ichi. Lattices non commutatifs. C. R. Acad. Sci. Paris **236**, 1525–1527 (1953).

The author generalizes the definitions of lattices and modular lattices to the case of non-commutative union and intersection. Theorem: A generalized lattice B can be decomposed into a direct sum of a generalized modular lattice B_N and a non-modular generalized lattice B_S where B_i $(i = N, S)$ is the direct sum of minimal ideals I_λ^i and all the I_λ^N are ordinary lattices isomorphic to each other.

P. M. Whitman (Silver Spring, Md.).

Dekker, J. C. E. The constructivity of maximal dual ideals in certain Boolean algebras. Pacific J. Math. **3**, 73–101 (1953).

The existence of non-atomic maximal dual ideals in Boolean algebras is usually proved by non-constructive means, such as the axiom of choice. The author gives two definitions of the notion of constructive dual ideal in the Boolean algebras V of all subsets, and E of all recursive subsets, of the set of non-negative integers. A dual ideal I is constructive in the first sense if the set of all its recursively enumerable elements is recursively enumerable; and in the second sense if it is the set of all supersets of the elements of a recursively enumerable set of sets. He then proves two principal results. The first is that a maximal dual ideal in E or V is constructive (in either sense) if and only if it is atomic. The second result is that in the Boolean algebra E of recursive sets, the two definitions of constructivity are equivalent; but in the Boolean algebra V of all subsets, constructivity in the second sense is stronger than constructivity in the first sense. *O. Frink.*

Krishnan, V. S. Les algèbres partiellement ordonnées et leurs extensions. II. Bull. Soc. Math. France **79**, 85–120 (1951).

[For part I see same Bull. **78**, 235–263 (1950); these Rev. **13**, 201.] The paper consists of two parts. In the first, the author develops a theory of ideals in partially ordered semigroups which parallels in its essential features a paper by W. Krull [Math. Ann. **101**, 729–744 (1929)] dealing with the case of rings without a finite chain condition. Since a distribution lattice is a special case of a partially ordered semi-group (but not of a ring), the author is able to obtain the results of Stone [Časopis Pěst. Mat. Fys. **67**, 1–25 (1937)] as a special case.

The second part of the paper is devoted to establishing a general theorem concerning the representation of abstract structures and the application of this theorem in a number of special cases. As the statement of the theorem depends on the author's definitions and postulates, it is not feasible to state it within the confines of a short review. Inherent in any relation (or operation) defined over a set of elements is

a partition of the set into equivalence classes of elements such that it is impossible to distinguish between the elements of any class by means of the given relation (or operation). Roughly speaking, in the second part of the paper, the author seeks greater generality by using inherent equivalence relations in place of equality in defining the structures and transformations admitted.

H. M. MacNeille (Providence, R. I.).

Dubreil-Jacotin, Marie-Louise. Propriétés algébriques des transformations de Reynolds. C. R. Acad. Sci. Paris **236**, 1136–1138 (1953).

Following Kampé de Fériet [Ann. Soc. Sci. Bruxelles. Sér. I. **63**, 165–180 (1949); these Rev. **11**, 336], the author considers Reynolds operators T on the ring of all functions on a domain E assuming a finite number of different values. The lattice of T-partitions of the domain is discussed. A necessary and sufficient condition is given for the intersection of all T-partitions to be itself a T-partition. (A partition is called a T-partition when T leaves invariant the characteristic function of every subset into which the partition divides E.) *G. Birkhoff* (Cambridge, Mass.).

Sikorski, R. Products of abstract algebras. Fund. Math. **39** (1952), 211–228 (1953).

$\mathfrak{A}$ being a class of algebras, an $\mathfrak{A}$ product (free product with respect to $\mathfrak{A}$) of algebras $A_i \in \mathfrak{A}$ is an algebra $B \in \mathfrak{A}$ generated by subalgebras $B_i \cong A_i$ such that any homomorphisms $h_i: B_i \to C \in \mathfrak{A}$ have a common extension $h: B \to C$. Sufficient conditions are given for the existence of $\mathfrak{A}$ products, and these conditions are applied to groups, Abelian groups, rings and commutative rings, as well as to various classes of lattices and Boolean algebras. It is not known whether free products of topological spaces always exist. However, modifying the notion of a free product by considering only a restricted class Φ of mappings, the author introduces the notion of an $(\mathfrak{A}, \Phi)$ product, which applied to topological spaces (with a suitable definition of Φ) yields their Cartesian products. *B. Jónsson.*

Mal'cev, A. I. On a class of algebraic systems. Uspehi Matem. Nauk (N.S.) **8**, no. 1(53), 165–171 (1953). (Russian)

Observing the existence of certain parallel results in group theory and ring theory, the author proposes a unification via the concept of a quasi-ring. There are to be two operations, the elements forming a (not necessarily abelian) group under $+$ and satisfying $x+y=y+x+D_1(x, y)$, $x(y+z)=xy+xz+D_2(x, y, z)$, $(x+y)z=xz+yz+D_3(x, y, z)$, $x\cdot 0=0\cdot x=0$. Here the D's are "polynomials" (sums of products of the elements or their inverses). The case of a ring is that where the D's are all 0. To cover the case of a group we define xy to be the commutator, set $D_1=xy$ and D_2, D_3 the appropriate expressions. The author takes up the following topics: homomorphisms, ideals, the descending central series, and quasi-rings satisfying identical relations. *I. Kaplansky* (Chicago, Ill.).

Evans, Trevor. Embeddability and the word problem. J. London Math. Soc. **28**, 76–80 (1953).

The main result of this paper is the theorem: There is a method for deciding the word problem for the (equationally defined) class of algebras $\mathfrak{A}$ if and only if there is a method for deciding the embedding problem for $\mathfrak{A}$. This completes the partial result of a previous paper [same J. **26**, 64–71 (1951); these Rev. **12**, 475] in which embeddability was shown to be a sufficient condition for solvability of the word problem. The present paper makes use of the results, methods, and notation of the previous paper. An example leads the author to raise again the question raised in the previous paper of the possibility of finding a set of axioms for abelian groups relative to which any incomplete abelian group is embeddable. *G. E. Bates.*

Asano, Keizo, und Ukegawa, Takasaburo. Ergänzende Bemerkungen über die Arithmetik in Schiefringen. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **3**, 1–7 (1952).

Let $\mathfrak{o}^*$ be a ring with unity 1 and $\mathfrak{o}$ be a subring of $\mathfrak{o}^*$ containing 1. Denote by $\mathfrak{f}$ the largest ideal of $\mathfrak{o}^*$ contained in $\mathfrak{o}$. If Σ is the set of all ideals $\mathfrak{a}$ of $\mathfrak{o}$ such that $(\mathfrak{a}, \mathfrak{f})=\mathfrak{o}$, and if Σ^* is similarly defined for $\mathfrak{o}^*$, then Σ and Σ^* are isomorphic as lattices and as multiplicative semigroups under the correspondence $\mathfrak{a} \to \mathfrak{o}^*\mathfrak{a}$. Assuming that $\mathfrak{o}$ and $\mathfrak{o}^*$ are equivalent maximal orders of a ring S, and that the usual finiteness conditions hold, the authors develop the arithmetic of $\mathfrak{o}$ relative to $\mathfrak{f}$. A typical result is as follows: each $\mathfrak{o}$-ideal $\mathfrak{a}$ has a unique representation $\mathfrak{a}=\mathfrak{p}_1 \cdots \mathfrak{p}_r \mathfrak{a}_0$ where the $\mathfrak{p}_i$ are regular prime ideals and $\mathfrak{a}_0$ is a divisor of some power of $\mathfrak{f}$.

R. E. Johnson (Northampton, Mass.).

Johnson, R. E. Representations of prime rings. Trans. Amer. Math. Soc. **74**, 351–357 (1953).

This is a continuation of the study of prime rings started by the author in a previous communication [Duke Math. J. **18**, 799–809 (1951); these Rev. **13**, 618]. It is assumed that there exists a mapping $I \to I^*$ of the set of the right (left) ideals of a prime ring R onto a subset $\mathfrak{R}$ (resp. $\mathfrak{L}$) of right (left) prime ideals satisfying certain closure relations and the condition that $\mathfrak{R}$ (resp. $\mathfrak{L}$) is atomic, i.e., it contains nonzero minimal elements called atoms (compare the above mentioned paper or its review). A subset $\mathfrak{R}$ (resp. $\mathfrak{L}$) of this kind is called a right (left) structure, and it is assumed in this paper that the prime ring R has both a right and a left structure.

For a right R-module M, a submodule N, and a subset A of M the right quotient $N:A$ is defined by $N:A=\{x; x \in R, Ax \subseteq N\}$. A right R-module M is called prime if $0:A=0$ for any submodule A of M. A submodule M' of M is called prime if $M-M'$ is a prime R-module. A prime R-module M is called admissible relative to a right structure $\mathfrak{R}$ if $0:x \in \mathfrak{R}$ for every $x \in M$. In the above mentioned paper the author has shown that if for a submodule N of M one defines the closure N^* of N by putting $N^*=\{x; x \in M, [N:x]^*=R\}$, then N^* is a prime submodule of M and the mapping $N \to N^*$ has properties analogous to those postulated for the mapping $I \to I^*$. The author says that the set of all N^* constitutes a right structure of M, induced by the right structure $\mathfrak{R}$ of R. Similarly he defines for a left R-module M the left structure of M induced by the left structure $\mathfrak{L}$ of R.

For a fixed atom N of $\mathfrak{R}$ there exists an atom M of $\mathfrak{L}$ so that $M \cdot N \neq 0$. The author shows that the nonzero ring $K=M \cap N$ of R is an integral domain that possesses a quotient division ring D. By attaching to K the trivial structure consisting of the two elements 0 and K and by considering N as an admissible (K, R)-module, he obtains the induced left structure $\mathfrak{N}$ of N, consisting of all closed K-submodules of N. The right structure of N induced by R is a trivial one since N is an atom of $\mathfrak{N}$. Similarly M has a nontrivial right structure $\mathfrak{M}$ induced by K. He shows that $A \in \mathfrak{N}$ if and only if $A=L \cap N$ for some L in $\mathfrak{L}$, and that the lattices

$\{\mathfrak{L}; \subseteq, \cup, \cap\}$ and $\{\mathfrak{N}; \subseteq, \cup, \cap\}$ are isomorphic under the correspondence $L \to L \cap N$. A dual result holds for $\mathfrak{R}$ and $\mathfrak{M}$. This is a generalization of an analogous well-known theorem concerning primitive rings with a nonzero socle. The author's main result is that in a sense specified below the ring R acts as an n-fold transitive ring of K-linear transformations on N for any integer n not exceeding the K-dimension of N: He shows that if $x_1, x_2, \cdots, x_n$ is any set of K-linearly independent elements of N and $y_1, y_2, \cdots, y_n$ any set in N, then there exists an element a of R and a nonzero element k of K such that $x_i a = k y_i$, $i = 1, \cdots, n$. *J. Levitzki.*

Levitzki, Jakob. On the rank of commutativity. Riveon Lematematika **6**, 1–14 (1953). (Hebrew. English summary)

Let $a_1, \cdots, a_k$ be elements of a ring S and set

$$[a_1, \cdots, a_k] = \sum \pm a_{i_1} \cdots a_{i_k},$$

the sum extending over the $r!$ permutations of $1, 2, \cdots, r$, with the usual sign convention. The least k for which $[a_1, \cdots, a_k] = 0$ holds identically in S is called the rank of commutativity of S, $d(S)$. This notion was introduced by F. W. Levi (who defined it as $d(S) - 1$) [J. Indian Math. Soc. (N.S.) **11**, 85–88 (1947); these Rev. **10**, 6]. Theorem 1. If A is an algebra over a field K of dimension m, then $d(A) \leq m+1$. Theorem 3. If N is the radical of N and $i(N)$ its index of nilpotency, then $d(A) \leq d(A/N) i(N)$, with the convention $d(A/N) = 1$ if $A = N$. Theorem 5. If A is the direct sum of the ideals A_i, then $d(A) = \max\{d(A_i)\}$. Theorem 6. If A is the algebra of $n \times n$ matrices over a field, then $d(A) = 2n$. This result, the principal one of the paper, was proved in another way by Amitsur and the author [Proc. Amer. Math. Soc. **1**, 449–463 (1950); these Rev. **12**, 155]. *I. S. Cohen* (Cambridge, Mass.).

Curtis, Charles W. A note on noncommutative polynomials. Proc. Amer. Math. Soc. **3**, 965–969 (1952).

A ring S is an extension of type O of a ring R in case: (a) R and S have the same identity element; (b) S contains an element x not in R such that, for each r in R, $rx = xT(r) + D(r)$ for $T(r), D(r)$ in R; (c) R and x generate S; (d) $\sum x^i r_i = 0$ implies $r_i = 0$. A necessary and sufficient condition, due to Ore, that a (noncommutative) integral domain R have a uniquely determined right quotient division ring is that R satisfy the following condition (M): Any two nonzero elements of R have a nonzero common right multiple. It is shown in this paper that, if R satisfies condition (M), then so does any extension S of type O, and consequently that condition (M) is preserved under an arbitrary number of extensions of type O. This theorem is applied to prove that the Birkhoff-Witt algebra A of a solvable Lie algebra of characteristic 0 satisfies condition (M), so that A has a uniquely determined right quotient division ring. The author points out that this result has subsequently been announced by D. Tamari [see the paper reviewed below] for the Birkhoff-Witt algebra of an arbitrary Lie algebra.

R. D. Schafer (Starrs, Conn.).

Tamari, Dov. On the embedding of Birkhoff-Witt rings in quotient fields. Proc. Amer. Math. Soc. **4**, 197–202 (1953).

Let R be a ring, Z its center, X a set of symbols and $R[X]$ a ring of Lie polynomials, i.e., $ax = xa$ for $a \in R$, $x \in X$ and $x_i x_j - x_j x_i = \sum_k \binom{ij}{k} x_k$, $\binom{ij}{k} \in Z$. For a fixed linear order of the indices of the x the author first shows, following essentially G. Birkhoff [Ann. of Math. (2) **38**, 526–532 (1937)] that each element of $R[X]$ has at least one standard form $\sum a_{\cdots n_i \cdots n_j \cdots} \cdots x_i^{n_i} \cdots x_j^{n_j} \cdots$ with $i < j$.

The ring $R[X]$ is called locally finite if any finite subset Y of X is contained in a finite subset $\bar{Y}$ of X such that $R[\bar{Y}]$ is also a (sub)ring of Lie polynomials. The author's main result is: If $R[X]$ is locally finite, and if R is a right regular integral domain, i.e., any two nonzero elements in R have a nonzero R.C.M. (=right common multiple), then any two nonzero elements P and Q of $R[X]$ have a formally nontrivial R.C.M., i.e., for some S and T we have $PS = QT = U$, where the standard form of U has at least one nonzero coefficient. The author reduces the proof of this theorem to determining nontrivial solutions for a system of right linear homogeneous equations where the number of unknowns (in this case the coefficients of S and T) exceeds the number of equations. The author shows that for a ring R without divisors of zero such solutions exist if and only if R is right regular. If the representation of any element of $R[X]$ by a standard form is unique, which is, e.g., the case when $R[X]$ is the universal enveloping algebra of a Lie algebra, one has $U \neq 0$. Hence, in this case $R[X]$ is right regular and may be imbedded by a well-known construction of Ore in a right quotient division ring. A special case of this result, i.e., when $R[X]$ is the universal algebra of a finite-dimensional Lie algebra of characteristic $p \neq 0$, has been previously proved by Jacobson [Amer. J. Math. **74**, 357–359 (1952); these Rev. **13**, 814]. Another special case concerning the universal algebra of a solvable Lie algebra of zero characteristic was recently announced by C. W. Curtis [see the paper reviewed above].

J. Levitzki (Jerusalem).

Pernet, Roger. Conjugaison et fibration dans les algèbres normales. I, II. C. R. Acad. Sci. Paris **236**, 1325–1327, 1403–1405 (1953).

Le langage utilisé par l'auteur de ces notes est tellement imprécis qu'il est à peu près impossible de savoir de quoi il est question. Aux rares endroits où il est fait allusion à des notions ou résultats classiques, on peut se demander si l'auteur les a compris. Il semble croire, par exemple, que le théorème classique de Wedderburn sur la décomposition d'une algèbre en somme d'une algèbre semi-simple et de son radical est valable sans hypothèse restrictive, ou que, pour une algèbre semi-simple somme directe d'algèbres simples A_m, il existe un surcorps du corps de base tel que les A_m aient toutes ce surcorps comme centre (à moins que l'auteur ne fasse aucune distinction entre une algèbre et l'algèbre obtenue par extension convenable du corps de base, ce qu'il est bien impossible de deviner d'après le contexte!). On pourrait aussi croire, d'après des théorèmes énoncés dans la seconde Note, qu'une algèbre extérieure R est telle que $R^2 = 0$! Le but général des Notes paraît être d'associer à une algèbre A une algèbre dite "satellite", qui serait une algèbre de quaternions lorsque A est une algèbre de matrices d'ordre pair. *J. Dieudonné* (Evanston, Ill.).

Krull, Wolfgang. Über unendliche algebraische Erweiterungen bewerteter Körper. Rend. Circ. Mat. Palermo (2) **1**, 164–169 (1952).

Let R be a normal, separable extension, not necessarily finite, of the valued field R_0. It is indicated how various results in the ramification theory can be obtained from the analogous results for finite extensions. In addition, the following is proved: Let R_t and R_r be the inertial field and ramification field respectively, Γ_t and Γ_r their respective value groups. Then the Galois group of R_r over R_t is iso-

morphic to the character group of Γ_v/Γ_t. [The ramification theory for infinite extensions is also discussed in Schilling, The theory of valuations, Amer. Math. Soc. Survey, no. 4, New York, 1950 [these Rev. 13, 315]; see chapter 3, and in particular, the last few lines of §2, p. 67.] *I. S. Cohen.*

Osima, Masaru. Some studies on Frobenius algebras. Jap. J. Math. 21 (1951), 179–190 (1952).

Some new results concerning Frobenius and related algebras are obtained and new proofs for some known theorems are given. For instance, the author gives conditions that a Frobenius algebra is weakly symmetric or almost symmetric [cf. G. Azumaya, same J. 19, no. 4, 329–343 (1948); these Rev. 11, 316–872]. Further, let A be a Frobenius algebra or a symmetric algebra or an almost symmetric algebra and let $\mathfrak{z}$ be a two-sided ideal of A. Various theorems are given which give necessary and sufficient conditions that $A/\mathfrak{z}$ be again of one of the three types. Let A be a Frobenius algebra and let φ be its Nakayama automorphism which is determined modulo an inner automorphism. With every automorphism σ of A, the author associates a regular element b_σ of A which satisfies the following conditions

$$a^{\sigma^{-1}\varphi\sigma} = (b_\sigma a b_\sigma^{-1})^\varphi \quad \text{for} \quad a \in A;$$
$$b_{\sigma\tau} = b_\sigma b_\sigma{}^\tau = b_\sigma{}^{\varphi\tau\sigma^{-1}} b_\tau$$

for two automorphisms σ, τ of A. *R. Brauer.*

Lang, Serge. The theory of real places. Ann. of Math. (2) 57, 378–391 (1953).

Real (i.e., real field (in the sense of Artin and Schreier [Abh. Math. Sem. Hamburg. Univ. 5, 85–99 (1926)]) (and ∞) valued) places of a field are studied. Firstly, a field with a real place is real. For any ordered real field K and any subfield F of K, a (real) place over F, called canonical, exists, which possesses as its valuation ring the totality of elements not infinitely large over F. A subfield of K is called archimedian over F, if it contains no element infinitely large over F. If F is maximally archimedian (over itself) in K, the canonical place of K over F is algebraic (i.e., algebraically valued) over F and may be extended to a canonical place of the real closure of K. A certain uniqueness theorem is proved, which is made use of in proving the "extension theorem" asserting that if P is a real closed field any P-valued place of a field K may be extended to a P-valued place of a real closure of K.

Further, if K is a real function field (in a finite number of variables) over a real closed constant field P, there exists a (real) place of K rational over P. A refinement of this theorem yields the following generalization of Artin's [ibid. 5, 100–115 (1926)] solution of Hilbert's problem: If K is a real function field over a real constant field F and if for every 0-dimensional real place ϕ of K/F either $\phi(\xi) = \infty$ or >0 for any ordering of $\phi(K)$, then ξ is a sum of squares in K. On considering forms of odd degrees only, an analogy to the theory of quasi-algebraic closure [Lang, Ann. of Math. (2) 55, 373–390 (1952); these Rev. 13, 726] is developed; the theory may be applied to real fields, since r forms of odd degrees in n common variables over a real closed field P have a common zero in P when $n > r$.

T. Nakayama (Nagoya).

Kasch, Friedrich. Über die Riccatische Differentialgleichung in Körpern der Charakteristik p. Arch. Math. 4, 17–22 (1953).

Suppose that K is a field of odd characteristic with a differentiation D in the sense of F. K. Schmidt and A. Jaeger [see, for example, Jaeger, Monatsh. Math. 56, 181–219 (1952); these Rev. 14, 530]. The author uses the fact that a linear differential equation $A(D)y = 0$ has a non-trivial solution in K if the differential polynomial $A(D)$ is a divisor of zero in the ring of D-differential polynomials over K, in order to solve the Riccati differential equation in such a field. *O. F. G. Schilling* (Chicago, Ill.).

Jordan, P. Algebraische Betrachtungen zur Theorie des Wirkungsquantums und der Elementarlänge. Abh. Math. Sem. Univ. Hamburg 18, 99–119 (1952).

The first half of this paper treats the algebraic aspects of the standard quantum mechanical formalism from three points of view: non-commutative algebra, projective geometry, and lattice theory. The lattice theory is that of orthocomplemented modular lattices as applied by G. Birkhoff and J. von Neumann to the logic of quantum mechanics. No novelty is claimed for the first part of the paper; the purpose is to suggest generalizations that may leave room for introducing the notion of elementary length into the theory. The author sees a great need for this, but knows no satisfactory theory embodying the concept of elementary length.

The second half deals with generalizations. The non-commutative algebras of Dirac q-numbers, matrices, and linear operations in Hilbert space can be generalized by using quaternions or the elements of the 8-unit non-associative Cayley algebra to replace complex numbers. This leads to the abstract notion of a formally complex algebra. The concept of Jordan algebra (named after the author) suggests that of power-associative algebra. The only new non-trivial example is that of the three-rowed matrices with elements in the Cayley algebra. This actually produces a generalization of the quantum mechanical formalism, including the probability concept.

The lattice theory generalization suggested involves giving up the commutative law for joins and meets in the theory of complemented modular lattices, leading to a theory of skew lattices. The need is also pointed out for a convenient analogue of ring addition or sum modulo 2, existing in Boolean algebras, in the case of complemented modular lattices.

Finally the author treats briefly the theory of right-rings, where addition is not necessarily commutative, and only one distributive law holds. The possible use of these interesting algebraic ideas in studying elementary length is not made clear. *O. Frink* (State College, Pa.).

Theory of Groups

Popova, Hélène. L'isotopie des logarithmétiques des quasi-groupes finis. C. R. Acad. Sci. Paris 236, 769–771 (1953).

Suppose that two quasigroups Q_1, Q_2 have the same order n and that their logarithmetics L_1, L_2 have the same order N. If Q_1, Q_2 are simple, have no proper subquasigroups, and are isotopic, then L_1, L_2 are isotopic with respect to addition. If L_1, L_2 can be represented by the same set of vectors with n components [cf. Popova, same C. R. 235, 1589–1591 (1952); these Rev. 14, 615], then L_1, L_2 are isomorphic with respect to multiplication; in particular, this is so when Q_1, Q_2 are plain and L_1, L_2 irreducible as defined in the paper cited. *I. M. H. Etherington* (Edinburgh).

Popova, Hélène. Sur la logarithmétique d'une boucle. C. R. Acad. Sci. Paris **236**, 1220–1222 (1953).

L denotes the logarithmetic of a finite quasigroup Q. If Q has one generator, then L is an additive loop if and only if Q is a loop. If L is an additive loop, then Q can be divided into disjoint classes each of which is a union of loops having the same unit element. L cannot be both an additive loop and a multiplicative group, except trivially when Q is idempotent. *I. M. H. Etherington* (Edinburgh).

Skolem, Th. A theorem on some semi-groups. Norske Vid. Selsk. Forh., Trondheim **25** (1952), 72–77 (1953).

The author acknowledges that the theorem in question (on the embedding of a semi-group in a group) is known.

A. H. Clifford (Baltimore, Md.).

Croisot, R. Propriétés des complexes forts et symétriques des demi-groupes. Bull. Soc. Math. France **80**, 217–223 (1952).

P. Dubreil [Mém. Acad. Sci. Inst. France (2) **63**, no. 3 (1941); these Rev. **8**, 15] showed that if R is the principal equivalence relation associated with a strong, symmetric (${}_RW=W_R$ and ${}_RR=R_R=R$), and neat ("net") (${}_RW$ and W_R empty) complex H of a demi-group D, then D/R is a semi-group (demi-group obeying both cancellation laws). If, in addition, H is a subgroupoid of D (in which case H is called normal), D/R is a group. The present author shows that D/R is a group without the additional assumption that H be a subgroupoid of D. A similar improvement is made in the case when H is not neat. Then $W={}_RW=W_R$ is the zero element of D/R, and the kernel of D/R (the set of non-zero-divisors of D/R) is a group. A full discussion of the situation is given, with a number of examples.

A. H. Clifford (Baltimore, Md.).

Thierrin, Gabriel. Quelques propriétés des équivalences réversibles généralisées dans un demi-groupe D. C. R. Acad. Sci. Paris **236**, 1399–1401 (1953).

The author extends to the case of an arbitrary complex H of a demi-group D a number of properties of the generalized right reversible equivalence relation Σ_R established by P. Dubreil [Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 183–200 (1951); these Rev. **14**, 12] for the case when H is a subgroupoid of D. Let T be the saturation of H by Σ_R, i.e., the set of all elements of D congruent mod Σ_R to some element of H. Let $V=\bigcup_{h\in R}(H^2:h)_d$, where $(H^2:h)_d$ means the set of all $x\in D$ such that $hx\in H^2$. In general, $H\subseteq V\subseteq T$. If $H=V$, then H is called "intègre à gauche"; this is equivalent to "unitaire à gauche" when H is a subgroupoid of D. In this case $H=T$ also, and $\Sigma_R\subseteq R_R$ where R_R is the principal right equivalence associated with H. If H^2 is strong, then $V=T$. Let $L=\bigcup_{h\in R}(H:h)_d$ and let P be the saturation of L by Σ_R. ($L=V$ and $P=T$ when H is a subgroupoid of D.) Then P is a left unitary subgroupoid of D, and P/Σ_R is a demi-group in which every element is a left identity. A number of further results are stated; no proofs are given. *A. H. Clifford* (Baltimore, Md.).

Iséki, Kiyosi. Sur les demi-groupes. C. R. Acad. Sci. Paris **236**, 1524–1525 (1953).

The author remarks that certain results of N. H. McCoy [Amer. J. Math. **71**, 823–833 (1949); these Rev. **11**, 311] on prime ideals and the radical of an ideal in a general ring hold for ideals in a demi-group. *A. H. Clifford.*

Conrad, Paul F. Embedding theorems for abelian groups with valuations. Amer. J. Math. **75**, 1–29 (1953).

Let A be an abelian operator group with the left operator domain R and let Γ be a partially ordered set. The author defines a Γ-valuation of A by assigning to each element $a\neq 0$ of A a non-empty trivially ordered subset of Γ, called the set of values of a, subject to the requirement that if none of the values of a and b ($a, b\in A$) is greater than γ (greater than or equal to γ), then none of the values of $a\pm b$ and ra ($r\in R$) is greater than γ (greater than or equal to γ). An abelian operator group with a definite Γ-valuation is called a Γ-group. If, to each γ in Γ, there is assigned an abelian operator group $B(\gamma)$ (with respect to a common operator domain R), then the Γ-sum V of the groups $B(\gamma)$ is defined as follows: V is the totality of vectors $b=(\cdots, b_\gamma, \cdots)$ with $b_\gamma\in B_\gamma$ and $b_\gamma=0$ for all γ with the exception of a set which satisfies the ascending chain condition. Addition and multiplication by R are defined componentwise. The values of b are the maximal γ's with $b_\gamma\neq 0$. Thus V becomes a Γ-group. V contains obviously the restricted direct sum of the groups $B(\gamma)$, i.e., the totality of all vectors $b=(\cdots, b_\gamma, \cdots)$ where all but a finite number of the components are zero, and V is contained in the complete direct sum of the groups $B(\gamma)$. If, in particular, Γ is trivially ordered or inversely well ordered, then the Γ-sum V coincides with the complete direct sum of the groups $B(\gamma)$. The subgroup C of V is a c-subgroup if, for every γ in Γ and $b(\gamma)\neq 0$ in $B(\gamma)$, there exists an element c in C with value γ such that $c_\gamma=b(\gamma)$. An isomorphism σ of a Γ-group to a Γ-group is called a Γ-isomorphism if σ and σ^{-1} are value-preserving isomorphisms. The author proves the following embedding theorem: If A is a Γ-group and if the operator domain R of A is a skew field, then A is Γ-isomorphic to a c-subgroup of the Γ-sum $V(A)$ of the groups $B(\gamma)=A^\gamma/A_\gamma$ where $A^\gamma(A_\gamma)$ is the totality of elements in A none of whose values is greater than γ (greater than or equal to γ). Moreover, any two embeddings of A into $V(A)$ are essentially the same. The Γ-group A itself is equal to a Γ-sum if and only if it does not possess any proper c-extensions.

After studying Γ-automorphisms of a Γ-group, the author considers torsion-free abelian groups. If such a group is division-closed, then it may be considered as an operator group over the rational number field. By making use of the fact that every torsion-free abelian group may be embedded in a division-closed group, the author applies his theory to the general case. Any partially ordered abelian group may be considered as a Γ-group in a natural way. In this fashion a simple proof of Hahn's classical embedding theorem for ordered abelian groups, and various results that go beyond, are obtained.

T. Szele (Debrecen).

Itô, Noboru. On a theorem of H. F. Blichfeldt. Nagoya Math. J. **5**, 75–77 (1953).

Let G be a group of linear transformations in n variables of finite order g. H. F. Blichfeldt [Trans. Amer. Math. Soc. **4**, 387–397 (1903)] showed that if g is divisible by a prime p which is larger than $(n-1)(2n+1)$, then G contains a normal abelian p-subgroup. The author shows that if G is assumed to be soluble, a similar result holds if $p>n$ except when $p=n+1$ and p is a Fermat prime, i.e., n is a power of 2.

R. Brauer (Cambridge, Mass.).

Szélpál, I. Über die untere Grenze der Ordnung n-stufig nichtkommutativer Gruppen. Comment. Math. Helv. **27**, 73–74 (1953).

For positive n, L. Redei has defined a finite group to be non-commutative of the nth kind if it is not of the $(n-1)$th kind and has a proper subgroup of the $(n-1)$th kind but not of higher kind, the groups of the 0th kind being the commutative groups [same Comment. **20**, 225–264 (1947); these Rev. **9**, 131]. The present author proves that for positive n the lower bound of the orders of the non-commutative groups of the nth kind is $3 \cdot 2^n$. *G. Higman.*

Nagai, Osamu. Note on Brauer's theorem of simple groups. Osaka Math. J. **4**, 113–120 (1952).

Let G be a group of finite order g which satisfies the following condition: There exist elements of prime order p which commute only with their own powers. The order g then is of the form $g=p(p-1)(1+np)/t$ where $1+np$ is the number of p-Sylow subgroups and where t is the number of classes of conjugate elements of order p; $t \mid (p-1)$. The following results were obtained by the reviewer [Proc. Nat. Acad. Sci. U. S. A. **25**, 290–295 (1939), Ann. of Math. (2) **44**, 57–79 (1943); these Rev. **4**, 266; **6**, 334]. If G is its own commutator subgroup and if $n<(2p+7)/3$, then $G \simeq LF(2, p)$ or p is a Fermat or Mersenne prime $p=2^m \pm 1$ and $G \simeq LF(2, 2^m)$. Using the same method, the author shows that for odd t the result still holds if the inequality for n is replaced by $n<p+2$. [In an unpublished investigation, the reviewer has determined all groups of the type in question with $n \leqq p+2$ and arbitrary t. Besides the groups mentioned, only the group $LF(3, 3)$ of order 5616 and the simple permutation group in 11 letters of order 7920 appear.]

R. Brauer (Cambridge, Mass.).

Fox, Ralph H. On Fenchel's conjecture about F-groups. Mat. Tidsskr. B. **1952**, 61–65 (1952).

The author proves the remaining case of Fenchel's conjecture concerning F-groups (see S. Bungaard and J. Nielsen, Mat. Tidsskr. B. **1951**, 56–58 [these Rev. **14**, 15] for the statement of Fenchel's conjecture and the cases hitherto proved). *D. G. Higman* (Montreal, Que.).

Fox, Ralph H. Free differential calculus. I. Derivation in the free group ring. Ann. of Math. (2) **57**, 547–560 (1953).

This first paper is preliminary in nature and develops machinery which, as the author tells us in the introduction, will subsequently be applied to topological problems in 3-dimensional manifolds, and, in general, to the theory of groups given by generators and relations.

The author gives a detailed account of the integral group ring of a free group X and its derivations. In general, if JG is the integral group ring of the group G (that is, $J \otimes G$) a derivation of JG is any mapping D of JG into itself which satisfies the conditions: $D(u+v)=Du+Dv$, and $D(u \cdot v)=Du \cdot v^\circ + u \cdot Dv$. Here v° is the integer assigned to v by the homomorphism O: $JG \to J$ induced by the trivial homomorphism of $G \to 1$. The author gives a detailed account of the derivations of the group ring JX of a free group X with generators $(x)=(x_1, x_2, \cdots)$. An element of JX can be thought of as a polynomial $f(x)=\sum a_u u$, $u \in X$, $a_u \in J$ (almost all $a_u=0$) where u are the reduced words of X.

The first result of the paper shows that: to each generator x_j of X there corresponds a derivation $f(x) \to \partial f(x)/\partial x_j$ called the derivative with respect to x_j, which has the property $\partial x_k/\partial x_j=\delta_{jk}$. Furthermore, there is one and only one derivation $f(x) \to f'(x)$ mapping $x_1, x_2, \cdots$ into prescribed elements $h_1(x), h_2(x), \cdots$ of JX; it is given by

$$f'(x)=\sum \frac{\partial f(x)}{\partial x_j} h_j(x).$$

These derivatives and their identities are subsequently studied and finally reapplied to JX. As a sample of the results obtained we mention the following: The nth power $\mathfrak{X}^n$ of the kernel of the homomorphism O, determines the nth lower central group of X. $f(x) \in JX$ belongs $\mathfrak{X}^n$ if and only if all the derivatives of orders less than n vanish at $x=1$. JX has no divisors of zero (theorem originally due to G. Higman). *R. Bott* (Ann Arbor, Mich.).

Ellis, David. On metric representations of groups. Math. Mag. **26**, 183–184 (1953).

Let G be a finite or countable group. It is shown that G is isomorphic to a subgroup of the group of all rigid motions of a metric space built over G. In the finite case, a stronger result is due to R. Frucht [Compositio Math. **6**, 239–250 (1938)]: every finite group is the group of a suitable graph.

G. Birkhoff (Cambridge, Mass.).

***Murnaghan, F. D. On convenient systems of parameters for the rotation group and the unitary group.** Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Diciembre, 1951, pp. 65–70. Centro de Cooperación Científica de la Unesco para América Latina, Montevideo, Uruguay, 1952. (Spanish)

This paper gives in Spanish the substance of two former papers in English [Proc. Nat. Acad. Sci. U. S. A. **38**, 69–73, 127–129 (1952); these Rev. **13**, 820].

G. de B. Robinson (Toronto, Ont.).

Murnaghan, Francis D. The characters of the symmetric group. Anais Acad. Brasil. Ci. **23**, 141–154 (1951).

The S-function $\{\lambda\}=\{\lambda_1, \lambda_2, \cdots, \lambda_r\}$, where (λ) is a partition of n, is expressed in terms of the power sums S_r as

$$\{\lambda\}=\sum \chi_p^{(\lambda)} S_p / 1^{\alpha_1} \alpha_1! 2^{\alpha_2} \alpha_2! \cdots m^{\alpha_m} \alpha_m!$$

where $\chi_p^{(\lambda)}$ is the value of the symmetric group character corresponding to the partition (λ) for the class

$$p=(1^{\alpha_1} 2^{\alpha_2} \cdots m^{\alpha_m}), \quad \text{and} \quad S_p=S_1^{\alpha_1} S_2^{\alpha_2} \cdots S_m^{\alpha_m}.$$

Then

$$\frac{\partial\{\lambda\}}{\partial S_j}=\{\lambda_1-j, \lambda_2, \cdots, \lambda_r\}+\{\lambda_1, \lambda_2-j, \cdots, \lambda_r\}+\cdots + \{\lambda_1, \lambda_2, \cdots, \lambda_r-j\},$$

the S-functions on the right being interpreted when the parts are not in descending order according to Murnaghan's convention strictly as (h_{λ_s-s+t}), so that, e.g., if a pair of consecutive numbers p, q occur among the parts, these may be replaced by the pair $q-1, p+1$, provided that the expression is changed in sign. If for this pair of consecutive numbers $p=q-1$, or again if for this last number of the set is a negative number, then the expression takes the value zero.

For brevity, the above equation will be written

$$\frac{\partial\{\lambda\}}{\partial S_j}=\sum\{\lambda'\}.$$

Then if p denotes the class $(1^{\alpha_1} 2^{\alpha_2} \cdots m^{\alpha_m})$ of the symmetric group of order $n!$ and p' denotes the class $(1^{\alpha_1} \cdots j^{\alpha_j - 1} \cdots m^{\alpha_m})$ of the symmetric group of order $(n-j)!$, it follows that

$$\chi_p^{(\lambda)}=\sum_{(\lambda')} \chi_{p'}^{(\lambda')}.$$

This result is the basis of the author's calculations.

If $(\mu)\equiv(\mu_1, \mu_2, \cdots, \mu_i)$ is a partition of p and (λ) denotes the partition $(n-p, \mu_1, \cdots, \mu_i)$ of n, then the symmetric group characteristic $\chi_p^{(\lambda)}$ can be expressed as a polynomial in $\alpha_1, \alpha_2, \cdots, \alpha_m$ which depends only on the partition (μ) and does not involve n explicitly. It involves n implicitly since $\alpha_1+2\alpha_2+\cdots+m\alpha_m=n$. This expression is denoted by $[\mu]$.

Since, for $j>p$, $\partial\{\lambda\}/\partial S_j=\{n-p-j, \mu_1, \cdots, \mu_i\}$, it follows that $[\mu]$ is unaltered by the deletion of a cycle of order j, and thus is a function only of $\alpha_1, \alpha_2, \cdots, \alpha_p$, say $[\mu]=f(\alpha_1, \alpha_2, \cdots, \alpha_p)$. For $j\leqq p$ the operation $\partial/\partial S_j$ operating on $\{\lambda\}$ is equivalent to the finite difference $\Delta f/\Delta\alpha_j$ where this denotes the decrement in f which results from the replacement of α_j by α_j-1. Such considerations enable the function $[\mu]=F(\alpha_1, \cdots, \alpha_p)$ to be calculated with reasonable ease for any given partition (μ). The author gives explicit polynomials for $[\mu]$ for all partitions (μ) of 1, 2, 3, 4, 5, 6, 7. In effect, these results give the values of all symmetric group characters corresponding to partitions (λ) of n provided that the first part $\lambda_1\geqq n-7$.

D. E. Littlewood (Bangor).

Brauer, Richard. A characterization of the characters of groups of finite order. Ann. of Math. (2) **57**, 357–377 (1953).

Let $\mathfrak{G}$ be a finite group. The main theorem states that a complex-valued function $\theta(G)$ on $\mathfrak{G}$ is a generalized character (i.e., rational integral linear combination of characters) of $\mathfrak{G}$ if and only if (I) θ is a class function and (II) the restriction of θ to an elementary subgroup $\mathfrak{E}$ is always a generalized character of $\mathfrak{E}$; a group is called elementary when it is a direct product of a p-group and a cyclic group [author, Amer. J. Math. **69**, 709–716 (1947); these Rev. **9**, 268]. θ is an irreducible character of $\mathfrak{G}$ if and only if (III) the average of $|\theta|^2$ is equal to 1 and (IV) $\theta(1)\geqq 0$, as well as (I), (II) [cf. author, Proc. Internat. Congress Math., Cambridge, Mass., 1950, vol. II, Amer. Math. Soc., Providence, R. I., 1952, pp. 33–36; these Rev. **13**, 530]. The main theorem is proved actually in a more general form dealing with linear combinations of characters with integral elements in an (arithmetic) field of characteristic 0. The proof depends on an arithmetic behaviour of a certain matrix arising from the relationship between irreducible characters of $\mathfrak{G}$ and those of a (maximal) elementary subgroup, and is in a sense a simplification of the argument previously used by the author [Ann. of Math. (2) **48**, 502–514 (1947); these Rev. **8**, 503] in proving Artin's conjecture. This last theorem, of induced characters, is in fact derived here from the present main theorem. Also, a characterization of linear characters of $\mathfrak{G}$ is derived from the main theorem. Further results on linear characters are given, leading to theorems on the relationship of $\mathfrak{G}/\mathfrak{G}'$ with certain factor groups of subgroups, obtained independently, and directly, by D. G. Higman [unpublished]. The main theorem is applied, moreover, to prove a theorem of Frobenius [S.-B. Preuss. Akad. Wiss. **1901**, 1216–1230], which deals with a generation of a certain generalized character by means of an invariant subset $\mathfrak{K}$, and from which follows the fact that the number of elements G with $G^n\in\mathfrak{K}$ is divisible by $(n, (\mathfrak{G}:1))$. Further, a construction of a generalized character from a modular character and the theorem of Cartan invariants [author, Ann. of Math. (2) **42**, 53–61 (1941); these Rev. **2**, 215], asserting that the determinant of the Cartan invariants is a certain power of the corresponding prime p, are derived from the main theorem. Finally, the determinant divisors of the matrix of decomposition numbers are 1, and a generalized character vanishing for all p-singular elements is a rational integral linear combination of the (modular) characters of direct components of the modular regular representation. *T. Nakayama* (Nagoya).

Roquette, Peter. Arithmetische Untersuchung des Charakterringes einer endlichen Gruppe. Mit Anwendungen auf die Bestimmung des minimalen Darstellungskörpers einer endlichen Gruppe und in der Theorie der Artinschen L-Funktionen. J. Reine Angew. Math. **190**, 148–168 (1952).

Let $\mathfrak{G}=\{G, R, P, \cdots\}$ denote a group of order g and of exponent g_0. Suppose that $K_\mathfrak{p}$ is the field of all g_0th roots of unity over the rational p-adic field, with the valuation ring $l_\mathfrak{p}$. The author studies in this paper the p-adic ring of characters $\mathfrak{X}_\mathfrak{p}$ which is associated to the group G by extending to $l_\mathfrak{p}$ the coefficient system of positive integers in the set of all characters, with addition of characters defined by the direct sum and multiplication of characters defined by the Kronecker product of the associated representations. He presents anew in an independent self-contained setting many of the now classical results of R. Brauer. Technical simplifications are achieved by considering the elements $\{\sum_\chi a_\chi\chi,\ a_\chi\in l_\mathfrak{p},\ \chi$ irreducible characters$\}$ of $\mathfrak{X}_\mathfrak{p}$ as functions on G with values in $l_\mathfrak{p}$, $G\to\sum_\chi a_\chi\chi(G)\in l_\mathfrak{p}$ and introducing pointwise convergence $\xi^{(n)}\to\xi$ if $\lim_{n\to\infty}\xi^{(n)}(G)=\xi(G)$ for all $G\in\mathfrak{G}$. The orthogonality relations for characters imply then that $\mathfrak{X}_\mathfrak{p}$ and its $l_\mathfrak{p}$-submodules are closed in this topology. The indecomposable ideals B of $\mathfrak{X}_\mathfrak{p}$—blocks—with their associated idempotents $\epsilon^{(B)}$ are uniquely determined by the p-upper classes $\mathfrak{O}$ of $\mathfrak{G}$ which consist of all elements in $\mathfrak{G}$ on which $\epsilon^{(B)}$ assumes the value 1, and a block B is in turn the set all functions in $l_\mathfrak{p}$ which assume the value 0 on the complement $\mathfrak{G}-\mathfrak{O}$; on the other hand, $\mathfrak{O}$ is characterized as the set of all elements $G, G', \cdots$ in $\mathfrak{G}$ which have conjugate factors R, R' of orders prime to p ($\mathfrak{p}|p$), where $G=RP$, $G'=R'P'$ and where P, P' have p-power orders. The ideal B contains a unique maximal ideal V which is characterized as the set of functions whose values lie in the prime ideal $\mathfrak{p}$ of $l_\mathfrak{p}$; V is the intersection $W\cap B=W\epsilon^{(B)}$ of the set of all functions in $\mathfrak{X}_\mathfrak{p}$ with values in $\mathfrak{p}$, and $W/\mathfrak{p}\mathfrak{X}_\mathfrak{p}$ is the radical of $\mathfrak{X}_\mathfrak{p}/\mathfrak{p}\mathfrak{X}_\mathfrak{p}$ so that $\mathfrak{X}_\mathfrak{p}/W$ is the direct sum of fields isomorphic to the residue class field of $K_\mathfrak{p}$. Using the theory of induced representations it is shown that each $\mathfrak{O}$ contains exactly one p-regular class R (in the sense of R. Brauer, orders of the elements in R are relatively prime to p) and $\mathfrak{O}$ consists of those elements $\mathfrak{G}$ whose factors R lie in R. In this manner the preparation for the reduction of the structure of the blocks to those related to elementary subgroups is made; if $R\in\mathfrak{O}$ is p-regular, $\mathfrak{P}$ a Sylow subgroup of the normalizer N_R, then the group H generated by RP is called an elementary subgroup for B. It is shown in detail how B and the $\mathfrak{p}$-adic ring of characters Π of $\mathfrak{P}$ are connected. If E is the block with the idempotent $\epsilon^{(E)}$ for H which belongs to the p-upper class $E=RP$, then let E^* denote all functions ξ^* on G which are induced by functions ξ of E as $\xi^*(G)=(1/[H:1])\sum_{X\in\mathfrak{G}}\xi(X^{-1}GX)$. Furthermore, B is isomorphic to a subring $\mathfrak{T}$ of Π where $\xi\in B$ corresponds to $\tau\in\Pi$ with $\xi\epsilon^{(E)}=\epsilon^{(E)}\tau$, i.e., ξ coincides with τ on E. The author's approach emphasizes analogies with Hasse's p-adic theory of hypercomplex arithmetic. The main results may now be summarized as follows. I. Each block contains a unique prime ideal. II. The blocks are in 1–1 correspondence to the p-regular classes. III. Each block B is isomorphic to a subring $\mathfrak{T}$ of the character ring of a p-subgroup, in the definite manner indicated above. IV. The mapping of III

is determined by the structure of P and its imbedding in N_R. V. The correspondence is determined by the theory of induced representations. Furthermore, R. Brauer's induction principles are examined carefully, and the theorem of the latter concerning the complete splitting of the integral group ring by g_0th roots of unity, and the meromorphic character of Artin's L-series are treated.

O. F. G. Schilling (Chicago, Ill.).

Witt, Ernst. Die algebraische Struktur des Gruppenringes einer endlichen Gruppe über einem Zahlkörper. J. Reine Angew. Math. **190**, 231–245 (1952).

Suppose that χ is an absolutely irreducible character of the finite group $\mathfrak{G}$; to χ there belongs then a uniquely determined simple component $\mathfrak{A}$, with $\chi(\mathfrak{A}) \neq 0$, of the group ring $K\mathfrak{G}$ of $\mathfrak{G}$ over the algebraic number field K. The author solves completely the problem to determine the Schur index $[\mathfrak{A}]$ from the group table of $\mathfrak{G}$ and the values of χ. For this purpose he starts with an independent study of the theorem of R. Brauer on induced representations [see also the preceding review] whose coefficients lie in a division algebra instead of a field. References to R. Brauer's theory of modular characters are appended so as to show modifications of the proofs. It is shown, partly with applications of the p-adic methods indicated in the paper reviewed above, that each character of $\mathfrak{G}$ relative to a division algebra $\mathfrak{S}$ is an integral linear combination of characters which are induced by characters of irreducible representations of subgroups with metabelian images for the corresponding representations. Now let $\mathfrak{S}$ be a division algebra with the center K. Suppose that the irreducible representation $\mathfrak{D}/\mathfrak{S}$ with the character $\chi_{\mathfrak{S}}$ is determined by a minimal left ideal of the algebra $\mathfrak{A}\mathfrak{S}^{-1}$. Then the given absolutely irreducible character χ determines $\chi_{\mathfrak{S}}$ uniquely, while $\chi_{\mathfrak{S}}$ determines the set of conjugate characters of χ whose sum shall be denoted by $\operatorname{sp}\chi$. A comparison of ranks yields $\chi_{\mathfrak{S}} = [\mathfrak{A}\mathfrak{S}^{-1}][\mathfrak{S}] \operatorname{sp}\chi$. On the other hand, letting $\mathfrak{F}$ and $\mathfrak{H}$ stand for subgroups with respective absolutely irreducible φ, ψ characters of $\mathfrak{F}$ and $\mathfrak{H}$, respectively, the multiplicity of φ in the induced character of ψ shall be denoted by (φ, ψ). Similarly $(\varphi_{\mathfrak{S}}, \psi_{\mathfrak{S}}, \mathfrak{S})$ shall be defined. If then φ, ψ are absolutely irreducible with the corresponding algebras $\mathfrak{A}$ and $\mathfrak{B}$ in the group rings $K\mathfrak{F}$, $K\mathfrak{H}$, respectively, then

$$(\varphi_{\mathfrak{S}}, \psi_{\mathfrak{S}}, \mathfrak{S}) = [\mathfrak{B}\mathfrak{S}^{-1}][\mathfrak{A}\mathfrak{S}^{-1}]^{-1}(\varphi, \operatorname{sp}\psi).$$

This formula turns out to be extremely useful for the proof of the theorem which states, for a given absolutely irreducible character χ of $\mathfrak{G}$, the existence of an elementary absolutely irreducible subgroup character ψ (with values in K containing the values of all characters, so that furthermore the field of gth roots of unity is of p-power degree over K) so that $(\psi, \chi) \not\equiv 0 \pmod{p}$; here elementary absolutely irreducible subgroup characters are defined in terms of subgroups which are contained in the normalizer of the group generated by a p-regular element. It also follows that the algebras $\mathfrak{A}$ and $\mathfrak{B}$ which are determined by ψ and χ are similar. In this manner the Hasse local invariants $J(\mathfrak{A})$ are shown to be equal to those of special algebras $\mathfrak{B}$ which can be computed more readily. The local components of $\mathfrak{B}$ occur in the following class of algebras which are termed cyclotomic: Suppose that R_q is the rational q-adic field, L a cyclotomic extension of R_q, and Z an intermediate field. Let $\beta(S, T)$ be a factor set of roots of unity for the Galois group of L/Z. Let C denote the crossed product of L/Z with the factor set $\beta(S, T)$. The Hasse invariant $J(\mathfrak{C})$ can then be computed as the invariant of a transfer algebra $\mathfrak{c}$ with the splitting field L/R_q and the factor set $\alpha_{\sigma,\tau} = \prod_{\nu} \beta(\bar{\nu}\sigma\bar{\nu}\sigma^{-1}, \bar{\nu}\sigma\tau\overline{\nu\sigma\tau}^{-1})^{\bar{\nu}-1}$, where ν runs over a system of representatives of the group of L/R_q modulo the group of L/Z, the bar above the symbols denotes fixed choices of representatives. (The author refers for the proof of this statement to an unpublished paper; note in this connection results in cohomology theory by S. MacLane, E. Artin, and J. Tate.) Then $J(\mathfrak{c})$ is computed by an interesting reduction to rational division algebras (application of the theorem of arithmetic progression, resulting change of ramifications), and it is found for $q \neq 2$ (the case $q = 2$ being as usual slightly different) that $J(\mathfrak{c}) = -m/(q-1) \bmod 1$, where the numerator m is determined as follows. The field L contains a maximal purely ramified field of hqth roots of unity (h is also a power of q) V and a maximal unramified subfield W with $VW = L$. Let F denote the Frobenius automorphism of L/V, and denote by Γ the Galois group of L/W and suppose that σ is the automorphism of L/W which raises the hth roots of unity to the power s, $(s, q) = 1$. Then m is given by $\alpha_{\sigma,F}^{h} \alpha_{F,\sigma}^{-h} \prod_{\nu \neq 1} \alpha_{\nu,\sigma} \equiv s^m$ modulo the prime ideal of L. It may be mentioned that these reduction formulas are quite useful for problems which occur in relation to the construction of fields with prescribed metabelian groups.

O. F. G. Schilling (Chicago, Ill.).

Osima, Masaru. Some notes on the induced representations of a group. Jap. J. Math. **21** (1951), 191–196 (1952).

Let V be a representation of a group G of finite order and let D be a representation of a subgroup H. If V_0 is the restriction of V to H and if an asterisk indicates the representation of G induced by a representation of H, then $D^* \times V \simeq [D \times V_0]^*$. An extension of this result to representations of groups by semi-linear transformations is given. Besides other applications, a new proof of a theorem of Nakayama and Shoda on semi-linear representations of groups is obtained [cf. T. Nakayama and K. Shoda, same J. **12**, 109–122 (1936)].

R. Brauer.

Sakai, Shoichiro. A remark on Mautner's decomposition. Kōdai Math. Sem. Rep. **1952**, 107–108 (1952).

Let $g \to U(g)$ be a continuous unitary representation of a separable locally compact group G in a separable Hilbert space H. Let $H = \int H_t ds(t)$ be a direct integral decomposition of H which commutes with the operators $U(g)$ so that for each g we have $U(g) = \int U(g, t) ds(t)$. Here, for each g and t, $U(g, t)$ is a unitary operator in H_t. Mautner [Ann. of Math. (2) **52**, 528–556 (1950); these Rev. **12**, 157] has shown that for almost all t there exists a continuous unitary representation $g \to V_t(g)$ of G in H_t such that $V_t(g) = U(g, t)$ for almost all g in G. Now for each g the operators $U(g, t)$ are determined only for almost all t and it is natural to ask whether the choices can be made so that $V_t(g) = U(g, t)$ for all g in G. In another article [Proc. Amer. Math. Soc. **2**, 490–496 (1951); these Rev. **13**, 11] Mautner has given an affirmative answer to this question for the special case in which G is a Lie group and Godement, in reviewing this second article, has outlined an argument which avoids the Lie group hypothesis. The present note supplies another proof that the answer is yes in general. The author asserts that he found it before becoming aware of Godement's review and that it is "slightly different" from Godement's proof.

G. W. Mackey (Cambridge, Mass.).

Takahashi, Shuichi. A duality theorem for representable locally compact groups with compact commutator subgroup. Tôhoku Math. J. (2) **4**, 115–121 (1952).

Let G be a locally compact group which has a compact normal subgroup whose quotient group is Abelian. Suppose also that G is maximally almost periodic. For such groups G the author proves a theorem which includes as special cases both the Pontryagin duality theorem for locally compact Abelian groups and the Tannaka duality theorem for compact groups. The dual $\hat{G}$ of G, defined as the set of all finite-dimensional unitary representations of G, is locally compact under a natural topology. The dual of $\hat{G}$, defined similarly, becomes a locally compact group into which G has a natural continuous imbedding. The author's theorem asserts that this imbedding is an isomorphism and homeomorphism onto. The proof makes use of both the Pontryagin and Tannaka theorems.

L. H. Loomis (Cambridge, Mass.).

Struble, Raimond A. Almost periodic functions on locally compact groups. Proc. Nat. Acad. Sci. U. S. A. **39**, 122–126 (1953).

Let G be a locally compact group with left invariant Haar measure μ. A one-parameter family $\{U_t : t>0\}$ of open sets with compact closures is called a left sampler family for G if it satisfies the following two conditions: (1) For every $\epsilon>0$ and every $y \in G$ there exists $a>0$ such that $\mu(yU_t \Delta U_t) < \epsilon\mu(U_t)$ for all $t>a$, where $A\Delta B$ is the symmetric difference $A'B \cup B'A$. (2) For every $c>0$ and $\epsilon>0$, there exists $b>0$ such that for every $t \geqq b$ there exists a finite disjoint set of left translates of U_s whose union differs from U_t (symmetric difference again) on a set of measure less than $\epsilon\mu(U_t)$. The author proves that if $\{U_t\}$ is a left sampler family for G, then $\lim_{t\to\infty} (1/\mu(U_t))\int_{U_t} f d\mu$ exists and is the von Neumann mean of f for every measurable almost periodic function f (in the sense of von Neumann). He also shows that a continuous function on G is von Neumann almost periodic if and only if it is Bohr almost periodic.

L. H. Loomis (Cambridge, Mass.).

NUMBER THEORY

Palamà, Giuseppe, e Poletti, L. Tavola dei numeri primi dell'intervallo 12 012 000–12 072 060. Boll. Un. Mat. Ital. (3) **8**, 52–58 (1953).

This list of primes between the limits indicated in the title contains 3684 entries.

D. H. Lehmer.

Kaprekar, D. R. Problems involving reversal of digits. Scripta Math. **19**, 81–82 (1953).

Paasche, Ivan. Ein neuer Beweis des Moessnerschen Satzes. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1952**, 1–5 (1953).

Salié, Hans. Bemerkung zu einem Satz von A. Moessner. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1952**, 7–11 (1953).

The theorem referred to was stated by Moessner [same S.-B. **1951**, 29 (1952); these Rev. **14**, 353] and proved by Perron [ibid. **1951**, 31–34 (1952); these Rev. **14**, 353]. The first of the present notes gives a proof using generating functions. In the second note the theorem is generalized so as to start with an arbitrary sequence of numbers and the corresponding result is then proved by induction.

H. W. Brinkmann (Swarthmore, Pa.).

Thébault, Victor. A note on number theory. Amer. Math. Monthly **60**, 322–323 (1953).

Jarden, Dov. Existence of an infinitude of composite n for which $2^{n-1} \equiv 1 \bmod n$. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **43** (1950), 52 (1952).

Gloden, A. Solution of the congruences $2x^2+1 \equiv 0 \pmod{p^3}$ and $2x^2+1 \equiv 0 \pmod{p^4}$. Scripta Math. **19**, 88–90 (1953).

Korobov, N. M. The distribution of non-residues and of primitive roots in recurrence series. Doklady Akad. Nauk SSSR (N.S.) **88**, 603–606 (1953). (Russian)

Suppose that, for all positive integers x, $\psi(x)$ satisfies the recurrence relation

$$\psi(x) = a_1\psi(x-1) + a_2\psi(x-2) + \cdots + a_n\psi(x-n)$$

for $x>n$, where $a_1, a_2, \cdots, a_n$ are fixed integers and $a_n \neq 0$; also let $\psi(1), \psi(2), \cdots, \psi(n)$ be integers not all zero. Let p be a prime greater than n. Then $\psi(x)$ is called a recurrence function of order n modulo p if a_n and at least one of the integers $\psi(i)$ $(1 \leqq i \leqq n)$ is not divisible by p. The sequence $\delta_1, \delta_2, \delta_3, \cdots$ whose mth member δ_m is the least non-negative residue of $\psi(m)$ modulo p is called a recurrence series of order n. Recurrence series of order n are necessarily periodic and the least period of such a series is denoted by τ, so that $1 \leqq \tau \leqq p^n - 1$. Write $S(N) = \sum_{x=1}^{N} e^{2\pi i \psi(x)/p}$, for $N \leqq \tau$. It is proved that $|S(\tau)| \leqq p^{\frac{1}{2}n}$, $|S(N)| \leqq p^{\frac{1}{2}n}(1+n \log p)$. A number of other theorems concerning the distribution of mth power residues and non-residues and of primitive roots in recurrence series is stated without proof. Thus Theorem 5 states that, for any $\epsilon>0$, there exists a $c = c(\epsilon, n)$ such that, for every recurrence series of order n and period $\tau > \tau_0 = cp^{\frac{1}{2}(n+1)+\epsilon}$, there occurs a primitive root modulo p among every $N = [\tau_0]+1$ consecutive numbers of the series.

R. A. Rankin (Birmingham).

Obláth, Richard. Über einige unmögliche diophantische Gleichungen. Mat. Tidsskr. A. **1952**, 53–62 (1952).

It is established that the following equations have no integral solutions with $|x|>1$: $x^3-1=2y^3$, $x^3-1=3y^n$ $(n \geqq 2)$, $48x^4+1=y^2$, $12x^4+1=y^4$, $x^3+1=3y^2$, $x^4 \pm x^2+1=3y^2$. The first two of these equations (with $n=2$) were treated earlier by the author [Acta Math. Acad. Sci. Hungar. **1**, 321–322 (1950); these Rev. **13**, 625, 1140] but the present work avoids the use of quadratic fields.

I. Niven.

Cugiani, Marco. Le frazioni continue. Period. Mat. (4) **30**, 257–267 (1952); **31**, 44–61 (1953).

Expository paper.

Pennington, W. B. On Mahler's partition problem. Ann. of Math. (2) **57**, 531–546 (1953).

Mahler's problem is to find an asymptotic formula for the number of partitions of n into powers of a fixed integer $r>1$ [J. London Math. Soc. **15**, 115–123 (1940); these Rev. **2**, 133]. Mahler's result had a multiplicative error of the form $e^{O(1)}$; the exact asymptotic behaviour was determined by the reviewer [Nederl. Akad. Wetensch., Proc. **51**, 659–669 = Indagationes Math. **10**, 210–220 (1948); these Rev. **10**, 16].

The present author generalises the problem by considering sequences $\lambda_1<\lambda_2<\lambda_3<\cdots$ of positive numbers which are such that $N(u)$ (i.e., the number of λ's which are $\leqq u$) has the form $N(u)=a\log u+b+R(u)$, where $\int_{\lambda_1}^{u}R(v)v^{-1}dv=V(\rho^{-1}\log u)+o(1)$ $(u\to\infty)$; $V(x)$ is integrable, bounded, and periodic mod 1, and ρ is a positive constant. The asymptotic behaviour of $P(u)$ is sought, where $P(u)$ denotes the number of solutions of $r_1\lambda_1+r_2\lambda_2+\cdots<u$ in integers $r_\nu\geqq 0$.

The author applies a general Tauberian theorem of Ingham [Ann. of Math. (2) **42**, 1075–1090 (1941); these Rev. **3**, 166] to the formula

$$\int_0^\infty e^{-us}dP(u)=\exp\left\{\int_0^\infty sN(u)(e^{us}-1)^{-1}ds\right\}=g(s).$$

Ingham's theorem is of the following type: If $P(u)$ is increasing (in the wide sense), and if in every angle $|(\operatorname{Im}s)/(\operatorname{Re}s)|\leqq\Delta$, $\operatorname{Re}s>0$, the behaviour of $g(s)$ for $s\to 0$ is of a certain type, then the asymptotic behaviour of $P(u)$ $(u\to\infty)$ can be determined. The author shows that in every angle of the type just mentioned we have $\log g(s)=\frac{1}{2}a\log^2 s-b\log s+\varphi(-\log s)+o(1)$, where $\varphi(z)$ is periodic mod 1 and analytical in the strip $|\operatorname{Im}z|<\frac{1}{2}\pi$, and the Fourier coefficients of $\varphi(z)$ are expressed in terms of those of $V(x)$. All conditions of Ingham's theorem are satisfied. The final result is:

$$\log P(u)=\tfrac{1}{2}a(\log u-\log\log u-\log a)^2+(a-\tfrac{1}{2})\log u+(b+\tfrac{1}{2})(\log u-\log\log u-\log a)+\varphi(\log u-\log\log u-\log a)-\tfrac{1}{2}\log 2\pi+o(1).$$

At the end of the paper the author shows that his results can be specialized to Mahler's partition problem in the form given by the reviewer ($\lambda_\nu=\nu^r$, r fixed, $r>1$; r need not be an integer). *N. G. de Bruijn* (Amsterdam).

Ayoub, Raymond. On Rademacher's extension of the Goldbach-Vinogradoff theorem. Trans. Amer. Math. Soc. **74**, 482–491 (1953).

The author applies Vinogradov's method to the problem of the representation of a large number as the sum of three primes, each of which lies in a given residue-class to a fixed modulus. He obtains the same asymptotic formula as was found by Rademacher on the assumption of the generalized Riemann hypothesis (or something approaching it). No essentially new difficulties are encountered in the work.
H. Davenport (London).

Linnik, Yu. V. Some conditional theorems concerning the binary Goldbach problem. Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 503–520 (1952). (Russian)

Assuming the Riemann hypothesis for the function $\zeta(s)$, the author proves that

$$\int_{-\chi}^{\chi}\left(\sum_2^\infty\Lambda(n)e^{-n/N}e^{-2\pi in\vartheta}\right)^2 e^{2\pi iN\vartheta}d\vartheta=e^{-1}N+O(N(\log N)^{-\epsilon}),$$

where $\Lambda(n)$ has the usual meaning in prime number theory, N is a large integer, $\epsilon>0$ arbitrary, $\chi=(\log N)^{-3(1+\epsilon)}$. From this it is easily deduced that the interval $(N, N+(\log N)^{3+\epsilon})$ contains a number which is a sum of two odd primes. If the Riemann hypothesis is replaced by the weaker conjecture

$$N(\sigma, T)=O(T^{3(1-\sigma)}\log^3 T),$$

the interval has to be enlarged to $(N, N+(\log N)^7)$. Similar results are formulated for intervals containing a Goldbach number in a given arithmetic progression. In this case the Riemann hypothesis has to be extended to all Dirichlet L-series corresponding to the modulus. *H. Heilbronn.*

Larsen, Otto. Epreuve numérique d'une supposition de P. Turán. Mat. Tidsskr. B. **1952**, 28 (1952).

Let $L_\nu(n)=\sum_{m=1}^{n}\lambda(m)m^{-\nu}$, where $\lambda(m)$ is Liouville's function ($\lambda(m)=(-1)^\mu$ when m is a product of μ equal or distinct primes). Turán conjectured that $L_1(n)>0$ $(n\geqq 1)$ [Danske Vid. Selsk. Mat.-Fys. Medd. **24**, no. 17 (1948); these Rev. **10**, 286]. Numerical evidence in support of this is here given for the range $1000<n\leqq 4528$. The author also reports that he has repeated previous calculations up to $n=1000$ and found minor discrepancies (in the fifth place of decimals). The least value found is $L_1(2837)=0.0002393$. Turán's conjecture is on much the same footing as Pólya's conjecture $L_0(n)\leqq 0$ $(n\geqq 2)$, and is open to similar doubts.
A. E. Ingham (Cambridge, England).

Carlitz, L. Some sums analogous to Dedekind sums. Duke Math. J. **20**, 161–171 (1953).

Working in the field of rational functions $GF(q, x)$, the author constructs analogues of the Dedekind sum $s(h, k)$ suggested by the representation

$$s(h, k)=1/(4k)+(1/k)\sum_\zeta(\zeta^{-1}-1)^{-1}(\zeta^h-1)^{-1},$$

where $\zeta\neq 1$ runs through the kth roots of unity. With $\omega_H(u)$ and $\psi(u)$ denoting functions discussed in earlier papers [same J. **1**, 137–168 (1935); **15**, 1001–1012 (1948); Trans. Amer. Math. Soc. **43**, 167–182 (1938); these Rev. **10**, 283], the function $Q_\nu(v)$ is defined by means of $(\psi(t)-v)^{-1}=\sum_{\nu=0}^{\infty}Q_\nu(v)t^\nu$, $v\neq 0$, and the sum

$$\sigma_r(H, K)=\sum_{\beta\neq 0}\frac{Q_r(\beta)}{\omega_H(\beta)},$$

where β runs through the roots of $\omega_K(u)=0$, becomes the analogue of the Dedekind sum and of the reviewer's generalization [same J. **17**, 147–157 (1950); these Rev. **11**, 641]. Several properties of the $\sigma_r(H, K)$ are derived, including the following reciprocity law: If $(H, K)=1$ and $q-1\mid r+2$, $r\geqq 0$, then we have

$$H\sigma_r(H, K)+K\sigma_r(K, H)=\sum_{i+j=r+2}b_ib_jH^iK^j-HK\sum_{i+j=r+2}b_ib_j.$$

The b_i are defined by $u/\psi(u)=\sum_{\nu=0}^{\infty}b_\nu u^\nu$ and can be thought of as analogues of the Bernoulli numbers.
T. M. Apostol (Pasadena, Calif.).

Kyhl, H. Some remarks on prime numbers. Mat. Tidsskr. A. **1952**, 79–81 (1952). (Danish)

It is shown that all the primes between $\sqrt{n}$ and n are the roots of an algebraic equation whose degree and integral coefficients depend only on n and the primes $\leqq\sqrt{n}$. The proof uses the cross-classification principle, from which is deduced an expression of the sum of the mth powers of the primes between $\sqrt{n}$ and n, in terms of those not exceeding $\sqrt{n}$. *W. J. LeVeque* (Ann Arbor, Mich.).

Kubilyus, I. P. On some problems of the geometry of prime numbers. Mat. Sbornik N.S. **31(73)**, 507–542 (1952). (Russian)

This is a development of the "analytical number-theory in n dimensions" introduced by Hecke on the basis of his zeta-functions associated with an algebraic number-field K of degree n. The existing theory is extended and strengthened by the use of the more powerful methods now available for estimating trigonometrical sums and the density of zeros

of zeta-functions. The results are too elaborate to summarise in detail, but they take the general form of asymptotic approximations to the number of prime ideal numbers $\mathfrak{p}$ for which $\mathfrak{p} \equiv \nu \pmod{\mathfrak{m}}$ ($\mathfrak{m}$ a given ideal of K, and ν a given ideal number prime to $\mathfrak{m}$) and $\mathfrak{p}$ (suitably interpreted as a point in n-dimensional Euclidean space) lies in a given region. Specialised forms of the theorems have implications about the distribution of rational primes. Thus the formula for the asymptotic distribution of Gaussian primes $\mathfrak{p}=k+il$ in sectors is obtained in a version that implies the existence of an infinity of rational primes p expressible in the form $p=k^2+l^2$ with rational integers k, l such that $l^2=O(p^\theta)$, where $\theta<1$ is a certain absolute constant. The numerical value found for θ depends on an estimate of the density of zeros of the relevant zeta-functions to the right of the critical line, and the method used here gives $\theta=25/32$. The paper has points of contact with results recently announced by Haselgrove [J. London Math. Soc. **26**, 273–277 (1951); these Rev. **13**, 438]. *A. E. Ingham.*

Eljoseph, Nathan. Arithmetic of dynames. Riveon Lematematika **6**, 15–23 (1953). (Hebrew. English summary)

The author investigates the number theory of the dynames $a+b\epsilon$ where a, b are real and $\epsilon^2=0$. He defines as integers of the first kind those numbers for which a is a rational integer, and as integers of the second kind those numbers of which both a and b are rational integers.

The mapping $a+b\epsilon \to a$ is a homomorphism, and for every polynomial $f(x_1+\epsilon y_1, \cdots, x_n+\epsilon y_n)=f(x+\epsilon y)$ we have

$$f(x+\epsilon y)=f_1(x)+\epsilon\left[\sum y_i \frac{\partial f_1}{\partial x_i}(x)+f_2(x)\right]$$

where the coefficients of f_1 are the real parts and the coefficients of ϵf_2 are the "imaginary parts" of the coefficients of f. Hence for integers of the first kind the diophantine equation $f(x+\epsilon y)=0$ has a solution only if the equation $f_1(x)=0$ has a solution in rational integers. Conversely, if $f_1(x)=0$ has a solution in rational integers and not all $\partial f_1/\partial x_i=0$ for that solution, then $f(x+\epsilon y)=0$ has a solution in integers of the first kind. In particular, the author proves unique factorization for integers of the first kind.

The situation is more complicated for integers of the second kind. There is in general no unique factorization or greatest common divisor. The author determines the primes and considers conditions under which there is a G.C.D.

E. G. Straus (Los Angeles, Calif.).

Carlitz, L. Some special equations in a finite field. Pacific J. Math. **3**, 13–24 (1953).

In the first part of this paper asymptotic formulas are obtained for the number of solutions of equations of the type $f_1(\xi_1)+\cdots+f_r(\xi_r)=a$, where the f_i are polynomials with coefficients in $GF(p^n)$, and $\xi_i, a \in GF(p^n)$. Use is made of the estimate of Mordell [Quart. J. Math., Oxford Ser. (1) **3**, 161–167 (1932)] for the exponential sum $S(f)=\sum_\xi e(f(\xi))$, where for $\beta \in GF(p^n)$, $e(\beta)=\exp(2\pi i t(\beta)/p)$ and $t(\beta)=\beta+\beta^p+\cdots+\beta^{p^{n-1}}$. In the second part exact formulas are obtained for the number of solutions of equations of the form $Q(\xi_1, \cdots, \xi_r)=f(\eta_1, \cdots, \eta_s)$, where Q denotes a quadratic form, while f is an arbitrary polynomial satisfying certain restrictions. In the latter problem the ξ_i are also permitted to be polynomials in $GF[p^n, x]$ of degree $<m$. *A. L. Whiteman* (Princeton, N. J.).

Carlitz, L. Note on a conjecture of André Weil. Proc. Amer. Math. Soc. **4**, 5–9 (1953).

Let V be a variety of dimension n without singular points defined over the finite field $GF(q)$, $q=p^n$ and N_m the number of rational points on V over the extended field $GF(q^m)$. A. Weil has made the conjecture that $\sum_{m=1}^{\infty} N_m u^{m-1}=d \log Z(u)/du$ where (1) $Z(u)$ is a rational function satisfying a simple functional equation and (2) having zeros and poles that satisfy some simple relations. The author shows that this conjecture holds when the variety is an equation of the form $Q(x_1, \cdots, x_s)=a$, where Q denotes a quadratic form. The conjecture also holds for an equation of the form $a_1 x_1^e y_1^f+\cdots+a_s x_s^e y_s^f=0$, $(e, f)=1$. In this case the variety contains singular points when $ef>1$. The conjecture (1) doesn't hold for an equation $a_1 x_1^e y_1^f z_1^g+\cdots+a_s x_s^e y_s^f z_s^g=0$, $(e, f, g)=1$. The author gives another example where the conjecture holds.

H. Bergström (Göteborg).

Roquette, Peter. Riemannsche Vermutung in Funktionenkörpern. Arch. Math. **4**, 6–16 (1953).

This paper comprises the outline of a proof of the Riemann hypothesis for function fields, whose methods center around properties of the twofold product of the function field by itself without specific reference to a general intersection theory. The author shows how a proof of the all-important positive-definiteness of A. Weil's trace function (which was introduced by the latter in his "Sur les courbes algébriques et les variétés qui s'en déduisent" [Hermann, Paris, 1948; these Rev. **10**, 262]) may be constructed with the arithmetic of suitable discrete rank one valuations for a basic tool. Suppose now that K and K' are fields of algebraic functions of one variable over a perfect field Ω with respect to which they are algebraically independent. In order to consider the algebraic correspondences between K and K' the author introduces (as Severi did in the Trattato di geometria algebrica, v. 1, pt. 1, Zanichelli, Bologna, 1926) the composite $\Delta=KK'$ which is a function field of 2 variables over Ω. In the field Δ valuations $w_{\mathfrak{M}}$ (in the sequel the symbol $\mathfrak{M}$ shall be used to denote the prime divisor and its attached homomorphism $z \to z\mathfrak{M}$, for $z \in \Delta$, $\{z\mathfrak{M} \neq \infty\}=\Delta\mathfrak{M}$, as well) are considered and with them the additive group of divisors which is generated by them over the integers. As in the work of Severi and Weil, for example, the prime divisors $\mathfrak{M}$ (or irreducible correspondences if one prefers a more geometric terminology) are classified on the basis of their effect on the component fields K and K'; constant prime divisors are those which are trivial on K' or K, that is, they are prolongations of prime divisors of K or K' to Δ, and the non-constant prime divisors which are trivial on both components. For the divisors of the latter type isomorphisms $K \to K\mu$ and $K' \to K'\mu'$ are then induced by residuation modulo $\mathfrak{M}$, and $\Delta\mathfrak{M}$ is a suitably defined dependent composite of $K\mu$ and $K'\mu'$. It is important for the subsequent work that the non-constant prime divisors are in one-to-one correspondence with the types of dependent composites of K and K' over Ω ($K\mu K'\mu'$ and $K\nu K'\nu'$ belong to the same type if there exists an isomorphism τ between them for which $\mu\tau=\nu$ and $\mu'\tau=\nu'$; the author indicates how these concepts have to be modified so as to have uniform statements for the constant prime divisors as well). Next, divisors $\mathfrak{a}$ and $\mathfrak{b}$ of Δ are placed in the same $\approx$-class if they differ by a divisor which is composed of constant prime divisors (see the $\equiv$-equivalence of Weil [loc. cit. p. 29]). Weil showed then that the Riemann hypothesis

is a consequence of the existence of a positive-definite linear functional $\sigma(\mathfrak{a}, \mathfrak{b})$ on the group of $\equiv$-classes ($\sigma(\mathfrak{a}, \mathfrak{a})>0$ if $\mathfrak{a}\not\equiv 0$). In order to demonstrate this key result the author develops a theory of intersection numbers which avoids explicit reference to the full geometric theory of products of correspondences and projections on an algebraic variety of Δ/Ω, though it must be said the differences of approach, compared with Weil, are not as deep as it may appear; occasionally they amount only to a difference in terminology. Using Deuring's arithmetic description of Severi's theory of correspondences [see Deuring, J. Reine Angew. Math. 177, 161–191 (1937); 183, 25–36 (1940); these Rev. 2, 246], and thus a generalization of Hasse's work for elliptic fields, for a given prime divisor $\mathfrak{p}$ of Δ the properties and existence of an arithmetic homomorphism of the group of divisors of Δ, which are prime to $\mathfrak{p}$, into the divisor group of $\Delta\mathfrak{p}$ are examined; the details of this existence proof by means of an explicit construction are very important (see also Deuring, Math. Z. 47, 643–654 (1942) [these Rev. 7, 362] for similar, though in nature quite different, problems). Suppose next that the divisor (z) of an element $z \in \Delta$ is defined as $(z)=\sum_{\mathfrak{q}} w_{\mathfrak{q}}(z)\mathfrak{q}$, $\mathfrak{a} \dashv \mathfrak{b}$ means $w_{\mathfrak{q}}(\mathfrak{a}) \leqq w_{\mathfrak{q}}(\mathfrak{b})$ for all $\mathfrak{q}$, and $\mathfrak{a}\sim\mathfrak{b}$ means that $\mathfrak{a}-\mathfrak{b}$ is the divisor of a function (this is the $\sim$-equivalence of Weil [loc. cit. p. 29]). Then the afore-mentioned homomorphism $\mathfrak{a}\to\mathfrak{a}\mathfrak{p}$ satisfies (1) $\mathfrak{a}+\mathfrak{b}=\mathfrak{c}$ implies $\mathfrak{a}\mathfrak{p}+\mathfrak{b}\mathfrak{p}=\mathfrak{c}\mathfrak{p}$, $\mathfrak{a} \dashv \mathfrak{b}$ implies $\mathfrak{a}\mathfrak{p} \dashv \mathfrak{b}\mathfrak{p}$, $\mathfrak{a}\sim\mathfrak{b}$ implies $\mathfrak{a}\mathfrak{p}\sim\mathfrak{b}\mathfrak{p}$; (2) $(z)\mathfrak{p}=(z\mathfrak{p})$ (the divisor of the image $z\mathfrak{p}$ in the field $\Delta\mathfrak{p}$); (3) if $\Delta\mathfrak{p}=K\pi K'\pi'$ and $\mathfrak{k}$, $\mathfrak{k}'$ are constant divisors (prime to $\mathfrak{p}$) from K, K', respectively, then $\mathfrak{k}\mathfrak{p}=\mathfrak{k}\pi$, $\mathfrak{k}'\mathfrak{p}=\mathfrak{k}'\pi'$ (with proper modifications in case that $K\to K\pi$ is not an isomorphism; in this connection see, especially for the properties of this homomorphism, Weil, loc. cit. pp. 31–35, for example, theorems 2 and 3, where its importance is made clear). As a next step the author defines the so-called residual product $\langle\mathfrak{a}, \mathfrak{p}\rangle$ of two divisors $\mathfrak{a}$ and $\mathfrak{p}$ (in this order!) as the divisor $N(\Delta\mathfrak{p}/K'\pi'; \mathfrak{a}\mathfrak{p})\pi'^{-1}$ of K'. (See proposition 2 on p. 38 of Weil, loc. cit.) Then the first property of (1) implies that $\langle\mathfrak{a}, \mathfrak{p}\rangle$ is linear in the first argument, while the third property of (1) implies that $\langle\mathfrak{a}, \mathfrak{p}\rangle\sim\langle\mathfrak{a}', \mathfrak{p}\rangle$ in K' if $\mathfrak{a}\sim\mathfrak{a}'$ in Δ, i.e., it enables one to relax the original assumption that $\mathfrak{a}$ be prime to $\mathfrak{p}$. The symbol $\langle\mathfrak{a}, \mathfrak{b}\rangle$ for composite $\mathfrak{b}$ is defined by linearity. Furthermore, the degree of $\langle\mathfrak{a}, \mathfrak{b}\rangle$ is denoted by $\chi(\mathfrak{a}, \mathfrak{b})$ and $\varphi(\mathfrak{a}, \mathfrak{b})$ is defined as

$$\deg(\Delta/K'; \mathfrak{a}) \deg(\Delta/K; \mathfrak{b}) + \deg(\Delta/K; \mathfrak{a}) \deg(\Delta/K'; \mathfrak{b}),$$

where $\deg(\Delta/K; \mathfrak{p})$, for example, denotes $[\Delta\mathfrak{p}: K\pi]$, using the terminology which was introduced above. Then

$$\sigma(\mathfrak{a}, \mathfrak{b}) = \psi(\mathfrak{a}, \mathfrak{b}) - \chi(\mathfrak{a}, \mathfrak{b})$$

is the desired functional of Weil (see p. 42, loc. cit.), which is easily seen to depend only on the $\sim$-classes of $\mathfrak{a}$ and $\mathfrak{b}$. Moreover, making suitable provisions for constant divisors, it follows that $\sigma(\mathfrak{a}, \mathfrak{b})$ is a one-valued function on the group of $\approx$-classes of Δ. The author's approach makes clearly the proof of the symmetric law $\langle\mathfrak{a}, \mathfrak{b}\rangle=\langle\mathfrak{b}, \mathfrak{a}\rangle$ an essential item of the theory. Because of the linearity and the behavior of divisors and multiplicities under coefficient extensions it suffices to prove $\langle\mathfrak{m}, \mathfrak{p}\rangle=\langle\mathfrak{p}, \mathfrak{m}\rangle$ for prime divisors $\mathfrak{m}$ and $\mathfrak{p}$ of degree 1. As noted at the outset, the divisors $\mathfrak{p}$ and $\mathfrak{m}$ determine essentially unique types of composites $\Delta\mathfrak{p}=K\pi K'\pi'$ and $\Delta\mathfrak{m}=K\mu K'\mu'$. One may then identify $K'\pi'$, $K'\mu'$ and K' so that $K\pi$ and $K\mu$ are subfields of K'. Then distinct prime divisors $\mathfrak{p}$ and $\mathfrak{m}$ determine distinct homomorphisms $K\to K\pi$ and $K\to K\mu$ into K'. For each prime divisor $\mathfrak{q}'$ of K' the author defines $\pi\mathfrak{q}'=N(K'/K\pi; \mathfrak{q}')\pi'^{-1}$ and $\mu\mathfrak{q}'=N(K'/K\mu; \mathfrak{q}')\mu'^{-1}$ in K. Next let $\{a\}$ be the set of elements in K which are integral for the valuations $\pi\mathfrak{q}'$ and $\mu\mathfrak{q}'$; it then follows that $\min w_{\mathfrak{q}'}(a\pi-a\mu)$ exists, and that it is zero for all but a finite number of valuations $\mathfrak{q}'$. Thus each minimum can be interpreted as the $\mathfrak{q}'$-adic value $w_{\mathfrak{q}'}(D'_{\pi,\mu})$ of a divisor $D'_{\pi,\mu}=D'_{\mu,\pi}$ of K'. Using the construction of the homomorphism $\mathfrak{a}\to\mathfrak{a}\mathfrak{p}$ it is indicated that this divisor is precisely $\langle\mathfrak{m}, \mathfrak{p}\rangle$; it is at this juncture that modifications of Weil's general intersection theory begin. Finally $\langle\mathfrak{p}, \mathfrak{p}\rangle\sim -W\pi$, where W is the canonical class of K, with $K'\pi'=K'$ and $K\pi\subseteq K'$ for initial assumptions. The actual proof for $\sigma(\mathfrak{a}, \mathfrak{a})>0$ involves the choice of an integral divisor $\sum_i \mathfrak{b}_i=\mathfrak{b}\approx\mathfrak{a}$, which contains no constant divisors from K' and has degree g (the genus of K) over K' (see Weil., loc. cit., proposition 3 on p. 43). If then the prime divisors $\mathfrak{b}_i$ have degree 1 over K', then there are given g homomorphisms $K\to K\beta_i\subseteq K'$. For these the author defines $d_\beta'=\sum_{i,\mu} D'_{\beta_i,\beta_\mu}$ and $N_\beta(W)=\sum_i W\beta_i$, and he asserts as a consequence of the symmetry property and the evaluation of $\langle\mathfrak{p}, \mathfrak{p}\rangle$ that $\langle\mathfrak{b}, \mathfrak{b}\rangle > d_\beta'-N_\beta(W)$. In this manner the positive-definiteness of the trace σ amounts to

$$\deg d_\beta' < 2 \deg(\Delta/K; \mathfrak{b})(2g-1).$$

The slightly stronger inequality

$$\deg d_\beta' \leqq 2 \deg(\Delta/K; \mathfrak{b})(2g-2)$$

can then be proved by means of the theorem of Riemann-Roch (for a useful comparison with Weil's work one may note that the author's $\mathfrak{b}$ corresponds to the former's X, pp. 53–54, $\deg(\Delta/K; \mathfrak{b})=\sum_i [K': K\beta_i]$ corresponds to Weil's $e=d'(X)$, and finally $\deg d_\beta' \leqq 2 \deg(\Delta/K; \mathfrak{b})(2g-2)$ is Weil's final inequality $\deg(Y\cdot\text{diagonal}) \leqq 2e(2g-2)$ where $X\circ X'=Y+e\cdot$diagonal on the middle of p. 54 in Weil, loc. cit.). The author sketches his proof, which in essence will correspond to portions of Weil's work on pp. 43–53, loc cit., and it is at this crucial mark that connections with general product varieties in one form or the other, though not necessarily as facts for abelian functions proper, will enter into the discussion. In the second part of the paper the Riemann hypothesis is proved, via the Frobenius automorphism of the coefficient field and its prolongation to the function field, essentially along the lines of Weil's original proof, loc. cit. pp. 68–72. *O. F. G. Schilling.*

Iwasawa, Kenkichi. On the rings of valuation vectors. Ann. of Math. (2) 57, 331–356 (1953).

Let K be a finite algebraic number field, or an algebraic function field of one variable over a finite constant field. The ring R of valuation vectors over K has the following properties: (1) R is a semi-simple commutative ring with unit element 1; (2) R is locally compact, but neither compact nor discrete; (3) R has a subfield K containing 1, such that K is discrete in R and the residue class space R/K is compact.

In the first half of his paper the author proves conversely that any topological ring having these properties is the ring of valuation vectors over a field K of the type described above. The main tool used is the notion of the norm $N(\sigma, G)$ of an automorphism σ of a locally compact group G, which is the factor by which the Haar measure in G is stretched by σ. It is multiplicative in σ; $N(\sigma, G)=N(\sigma, H)N(\sigma, G/H)$ if $\sigma H\subset H$; and $N(\sigma, G)=1$ if G is compact or discrete. If F is a nondiscrete locally compact field, then the function $N(\alpha, F)$ for $\alpha \in F^*$ gives the "normed" valuation of F. The intersection of all closed maximal ideals M of R is 0, and $N(x, R)=\prod_M N(x, R/M)$ for regular $x \in R$. In particular, if

$\xi \in K^*$, then from (3), $N(\xi, R)=1$, so that the valuations $N(\xi, R/M)$ of K satisfy the "product formula." From this it follows by more or less familiar methods that K is an arithmetic field and R its valuation vector ring. The next section contains a proof that valuation vector rings do enjoy properties (1), (2), and (3).

Finally, the author shows that the two central theorems of algebraic number theory can be proved directly from the topological properties of R. The first of these is the self-duality of R with respect to K, on which the analog of the Riemann-Roch theorem for number fields rests. If χ is any non-trivial additive character of R vanishing on K, then the map $x \to \chi(xy)$ is an isomorphism of R onto its character group, and $\chi(x\xi)=1$ for all $\xi \in K$ if and only if $x \in K$. The essential step in the proof is to show that the kernel of the map $x \to \chi(xy)$ is a compact ideal of R, and is therefore 0. The second theorem states the compactness of the group of idèle classes of norm 1, which is equivalent to the finiteness of class number and unit theorem. The idèle group J is the multiplicative group of regular elements of R. It is a locally compact group in the topology for which the convergence of a_i in J means the convergence of a_i and a_i^{-1} in R. The map $a \to N(a, R)$ is a homomorphism of J with kernel J_1, and the compactness of J_1/K^* is proved by a Minkowski-type argument using Haar measure.

Throughout the paper, the case of function fields over non-finite constant fields is treated in parallel, using linear compactness instead of compactness.

J. T. Tate (New York, N. Y.).

LeVeque, W. J. Geometric properties of Farey sections in $k(i)$. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 415–426 (1952).

By use of a diagram of spheres [Speiser, J. Reine Angew. Math. **167**, 88–97 (1932)] the boundaries of the Farey sections in the Gaussian field as defined by Cassels, Ledermann, and Mahler [Philos. Trans. Roy. Soc. London. Ser. A. **243**, 585–626 (1951); these Rev. **13**, 323] are given a simple construction; they are the projection of the center of a sphere which moves so as to be tangent to two spheres and the complex plane. This enables the author to simplify, clarify, and extend the earlier results and proofs.

L. Tornheim (Ann Arbor, Mich.).

LeVeque, W. J. Continued fractions and approximations in $k(i)$. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 526–535, 536–545 (1952).

An algorithm and some properties for a regular continued fraction expansion for complex numbers with partial quotients which are Gaussian integers are obtained by using results of the paper reviewed above. The convergents form a subset of the best approximations as defined by the Farey sections of the complex plane. Unfortunately the successive partial quotients sometimes depend not only on the preceding complete quotient but also on the preceding convergent; there is a discussion of when this occurs. This difficulty prevents introducing the notion of equivalent numbers. Finally, several probability theorems are obtained on distributions involving the partial quotients and the complete quotients of the continued fractions and the approximating numbers in the Gaussian field.

L. Tornheim (Ann Arbor, Mich.).

Jones, B. W. Correction to "An extension of Meyer's theorem on indefinite ternary quadratic forms". Canadian J. Math. **5**, 271–272 (1953).

G. L. Watson has shown by a counter-example that the results of the author's paper are not completely correct. This counter-example (indicated in the present paper and which seems not to have been published by G. L. Watson in any other place) is given by the form

$$f=2x^2+12y^2+byz+12z^2.$$

In order to make his results correct, the author adds to his Theorems 2, 4, 6 some supplementary conditions. After these modifications, the author's theorem appears as a generalization of that of Meyer (where the g.c.d. of Ω and Δ is 1 or 2), not to the case where the g.c.d. of Ω and Δ is free from square factors, but only to the case where this g.c.d. is prime (see the review of the author's paper [same J. **4**, 120–128 (1952)] in these Rev. **13**, 537).

M. Krasner (Paris).

Ollerenshaw, Kathleen. An irreducible non-convex region. Proc. Cambridge Philos. Soc. **49**, 194–200 (1953).

Let K be the region of points (x, y) for which $|x, y| \leqq 1$, $|x^2-y^2| \leqq 2$, and let $\Delta(K)$ denote the lower bound of the determinants of the admissible lattices of K. A star domain H is said to be irreducible if, for every star domain H' which is a proper subset of H, $\Delta(H')<\Delta(H)$. The author determines the irreducible star domain H within K for which $\Delta(H)=\Delta(K)$ and proves it to be unique. This is the first known example of a reducible star domain Q which contains a unique irreducible star domain P for which $\Delta(P)=\Delta(Q)$. Where $A(H)$ is the area of H, it is pointed out that $A(H)/\Delta(H)$ is less than the corresponding ratio for any other known star domain. A domain G is constructed which is not a star domain for which $A(G)/\Delta(G)$ takes the least known value of this quotient.

D. Derry.

Chabauty, Claude. Résultats sur l'empilement de calottes égales sur une périsphère de R^n et correction à un travail antérieur. C. R. Acad. Sci. Paris **236**, 1462–1464 (1953).

The author considers the packing of non-overlapping spherical caps of semi-angle θ on the surface Σ_n of the n-dimensional unit sphere. Let $\nu(\theta, n)$ be the maximum number of such caps that can be placed on Σ_n. The following inequalities are obtained:

$$\{\nu(\theta, n)\}^{1/n} \leqq \{1+\epsilon(n)\}\{\sqrt{2}\sin\theta\}^{-1},$$

valid, presumably, for $\theta \leqq \pi/4$ and not $\theta \leqq \pi/2$ as stated, and

$$\{\nu(\theta, n)\}^{1/n} \geqq \{1-\epsilon(n)\}(\sin 2\theta)^{-1},$$

for $\theta \leqq \pi/2$. Here $\epsilon(n)$ is a function which tends to zero with $1/n$. In the particular case $\theta=\pi/6$ it is deduced that the maximum number $\tau(n)$ of equal disjoint spheres which can touch a sphere of the same radius satisfies the inequalities

$$\tfrac{1}{2}-\epsilon(n) \leqq \{\tau(n)\}^{\frac{1}{n}} \leqq 2+\epsilon(n).$$

Several other results are stated and a correction is made to a previous paper [same C. R. **235**, 529–532 (1952); these Rev. **14**, 561]. Note: the right-hand side of (1) should read $(n+1)\omega_n$ and not $n+1$.

R. A. Rankin (Birmingham).

Mahler, K. On the lattice determinants of two particular point sets. J. London Math. Soc. **28**, 229–232 (1953).

If K is a domain in the plane, then its determinant $\Delta(K)$ is defined as the inf of the determinants of all lattices which do not contain inner points of K, apart from the origin. Consider the following regions: $K(|xy| \leqq 1)$; $R_t(|xy| \leqq 1, x \geqq t)$;

$K_t(|xy| \leqq 1, |y| \leqq t)$; S_t is the union of R_t and the strip $0 \leqq x \leqq t$ ($t>0$). From the facts that $\Delta(K)=\sqrt{5}$ (Hurwitz) and $\Delta(K_t)=\sqrt{5}$ [K. Mahler, Proc. Roy. Soc. London. Ser. A. **187**, 151–187 (1946); these Rev. **8**, 195], the author derives that $\Delta(R_t)=\sqrt{5}$. Furthermore, he shows that $\Delta(S_t)=\frac{1}{2}(3+\sqrt{5})$, which is deduced from a theorem of A. V. Prasad [J. London Math. Soc. **23**, 169–171 (1948); these Rev. **10**, 513], which asserts that to every real number ξ at least one fraction u/v can be found such that

$$|\xi - u/v| \leqq 2v^{-2}(3+\sqrt{5})^{-1}.$$

It is trivial that $\Delta(R_t)$ and $\Delta(S_t)$ do not depend on t (if $t>0$), as all regions R_t are equivalent under linear transformations of determinant 1, and the same holds for S_t. Nevertheless, it is surprising to notice the fact that under the regions with determinant 1 some are "arbitrary small", and some are "arbitrary large". *N. G. de Bruijn.*

Eichler, Martin. Note zur Theorie der Kristallgitter. Math. Ann. **125**, 51–55 (1952).

Let R_0 be an n-dimensional vector space over the rational field k_0, with a positive definite metric (scalar product) with the properties (1) $\xi\eta=\eta\xi \in k_0$, (2) $(x\xi)\eta=x(\xi\eta)$, (3) $(\xi+\eta)\zeta=\xi\zeta+\eta\zeta$, (4) $\xi^2>0$ for $\xi\neq 0$, where $\xi, \eta \in R$, $x \in k$. A crystal-lattice (lattice) in R_0 is a $\mathfrak{o}_0$-module $\mathfrak{J}$ of rank n (additive abelian group) with respect to the order (integral set) $\mathfrak{o}_0$ of all rational integers. If $\mathfrak{J}$ has mutually orthogonal sublattices $\mathfrak{J}_1, \mathfrak{J}_2, \cdots$, of ranks $r_1, r_2, \cdots, r_1+r_2+\cdots=n$, such that every $\iota \in \mathfrak{J}$ is uniquely expressible in the form $\iota=\iota_1+\iota_2+\cdots$, $\iota_\nu \in \mathfrak{J}_\nu$, then $\mathfrak{J}$ is said to be the direct sum $\mathfrak{J}=\mathfrak{J}_1+\mathfrak{J}_2+\cdots$, of $\mathfrak{J}_1, \mathfrak{J}_2, \cdots$. The author proves (Theorem 1) that if $\mathfrak{J}=\mathfrak{J}_1'+\mathfrak{J}_2'=\mathfrak{J}_1+\mathfrak{J}_2$, both sums direct, then $\mathfrak{J}_1=\mathfrak{J}_1\cap\mathfrak{J}_1'+\mathfrak{J}_1\cap\mathfrak{J}_2'$, and similarly for $\mathfrak{J}_2, \mathfrak{J}_1', \mathfrak{J}_2'$. From this follows (Theorem 2) that if

$$\mathfrak{J}=\mathfrak{J}_1+\mathfrak{J}_2+\cdots=\mathfrak{J}_1'+\mathfrak{J}_2'+\cdots,$$

both sums direct, with directly indecomposable summands, then the $\mathfrak{J}_\nu$ and $\mathfrak{J}_\nu'$ are identical apart from order. Bases of $\mathfrak{J}$, etc., relative to $\mathfrak{o}_0$ are employed in the proof of Theorem 1, and hence the proof does not go in general for an algebraic number field k. Also, an example shows that the definiteness (property 4) of the metric is essential. However, the author obtains Theorems 1 and 2 for a field k of degree h over k_0, and an arbitrary order $\mathfrak{o}$ in k (i.e., not necessarily the set of all integers of k), provided k has an involutory automorphism $x\to\bar{x}$, such that $x+\bar{x}$ and $x\bar{x}$ are totally real for all x in k. Then $\xi\to\bar{\xi}$, with $\overline{x\xi}=\bar{x}\bar{\xi}$, is an extension of the automorphism to an n-dimensional vector space R over k. The scalar product is required to satisfy (1) $\xi\eta=\overline{\xi\eta} \in k$, (2) $(x\xi)\eta=x(\xi\eta)$, (3) $(\xi+\eta)\bar{\zeta}=\xi\bar{\zeta}+\eta\bar{\zeta}$, (4) $\xi\bar{\xi}$ totally positive for $\xi\neq 0$. The author gets around the basis difficulty by extending R over k to a space of dimension hn over k_0, with scalar product $\xi\circ\eta=\mathrm{trace}_{k/k_0}(\xi\eta)=\mathrm{trace}_{k/k_0}(\eta\bar{\xi})$, which has properties (1), ···, (4) above. *R. Hull.*

Maxfield, John E. Normal k-tuples. Pacific J. Math. **3**, 189–196 (1953).

A k-tuple β is a k-tuple $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ of real numbers. The nth k-digit of the k-tuple to the base r is

$$b^{(n)}=(A_1^{(n)}, \cdots, A_k^{(n)})$$

where $A_s^{(n)}$ is the nth digit of the fractional part of α_s to the base r. A k-tuple is said to be simply normal to the base r if the number n_c of occurrences of the kth digit c in the first n k-digits of the fractional part of β has the property $\lim_{n\to\infty} n_c/n=1/r^k$ for each of the r^k possible values of c. A k-tuple β is said to be normal to the scale r if β, $r\beta$, $r^2\beta$, ··· are each simply normal to all the scales $r, r^2, \cdots$ where $r^s\beta=(r^s\alpha_1, \cdots, r^s\alpha_k)$. The author extends many of the theorems on normal numbers to normal k-tuples and proves several new theorems. Perhaps his most interesting theorem is the following. The k-tuple $\beta=(\alpha_1, \alpha_2, \cdots, \alpha_k)$ is normal if and only if $\sum_{i=1}^k h_i\alpha_i$ is a normal number for all integers $(h_1, h_2, \cdots, h_k) \neq (0, 0, \cdots, 0)$. *P. Erdös.*

Descombes, Roger. Sur un problème d'approximation non homogène. C. R. Acad. Sci. Paris **236**, 1401–1403 (1953).

Let s be a positive integer and for irrational ξ and integers a, b with $(a, b, s)=1$ put

$$(*) \qquad H(\xi, a, b)=\liminf |v(v\xi-u)|$$

taken over integers v, u with $u\equiv a$, $v\equiv b$ (s). The upper bound of $H(\xi, a, b)$ over ξ is independent of a, b and is denoted by $H(s)$. If s is even, then $H(2)=1$, $H(s)=\frac{1}{4}s(s-2)$ $(s>2)$ and it is non-isolated. If s is odd, then $H(s)$ is isolated; the values for $3 \leqq s \leqq 11$ are given and

$$H(s)=\tfrac{1}{4}(s^2-2s-1)(1-2/s)^{-1/2}$$

for $s \geqq 13$. The proofs depend on showing that for (*) one need consider only (u, v) related to the successive convergents in a certain way. *J. W. S. Cassels.*

Descombes, Roger. Sur un théorème classique d'Hurwitz. C. R. Acad. Sci. Paris **236**, 1460–1462 (1953).

For irrational ξ and integer s define

$$c_s(\xi)=\limsup |q(q\xi-p)|^{-1}$$

taken over integers p, q with $s \nmid q$. Let c_s, γ_s be the upper bound of $c_s(\xi)$ and its greatest point of accumulation, respectively. Then $c_2=\gamma_2=2$, but $c_s=5^{1/2}$ for $s>3$. For $s>2$ the value of γ_s depends on the index j of the first term in the Fibonacci sequence 0, 1, 1, 2, ··· divisible by s. If j is even, then $\gamma_s=1+3.5^{-1/2}$ and $c_s(\xi)$ takes no value between c_s and γ_s. If j is odd, then γ_s is given by a complicated expression depending on j and there are infinitely many ξ with $c_s(\xi)>\gamma_s$. Proofs depend on considering best "s-approximations" p_n/q_n with $s \nmid q_n$ and such that $|q\xi-p| \geqq |q_n\xi-p_n|$ if $|q| \leqq |q_n|$ and $s \nmid q$. *J. W. S. Cassels.*

***Skolem, Th. Some theorems on irrationality and linear independence.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 77–98. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

Of the many results obtained by the author we cite the following. (1) Let $f(x)=\sum_{n=0}^\infty [nx]/n!$ and let $x_1, x_2, \cdots, x_l$ be different positive numbers, such that 1 is not linearly dependent on them. Then the numbers $1, f(x_1), \cdots, f(x_l)$ are linearly independent. (Here "linear dependence" is always meant with respect to the field of rational numbers.) (2) Let $g, h_1, h_2, \cdots, h_m$ be integers >1 and let $h_1, h_2, \cdots, h_m$ be different. Then the m numbers $\sum_{n=0}^\infty g^{-h_\nu^n}$ $(r=1, 2, \cdots, m)$ are linearly independent. (3) Let g and h denote integers >1 and let $f(0), f(1), f(2), \cdots$ be integers, such that $0<f(0) \leqq f(1) \leqq f(2) \leqq \cdots$ and such that

$$\limsup \frac{1}{n} \log \frac{\log f(n)}{\log g} < \log h.$$

Then the number $\sum_{n=0}^\infty f(n)g^{-h^n}$ is transcendental.

The methods used are similar to the "Reihenmethode" in Koksma, "Diophantische Approximationen" [Springer, Berlin, 1936, p. 54].

By another method the author derives: (4) Let h and k be positive integers, $h>1$, let

$$f(x)=\sum_{n=0}^{\infty}h^{-\binom{n+1}{2}}k^{-nx^n}.$$

Moreover, let $k_1, k_2, \cdots, k_l$ be rational numbers $\neq 0$, such that none of the quotients k_i/k_j $(i\neq j)$ is equal to an integral power of h. Then $f(0)$ and the numbers $f^{(s)}(k_r)$ $(r=1, 2, \cdots, l;\ s=0, 1, \cdots)$ are linearly independent. (The reviewer remarks that this last result is closely related to a theorem of Tschakaloff [Math. Ann. **84**, 100–114 (1921)].)

J. Popken (Utrecht).

Korobov, N. M. On some problems of Čebyšev type. Doklady Akad. Nauk SSSR (N.S.) **89**, 397–400 (1953). (Russian)

It is known that, for given irrational α, the inequalities

$$0<x<ct,\quad |\alpha x-y-\beta|<\frac{1}{t}\quad (c=c(\alpha)),$$

have solutions in integers x and y for arbitrary real β and any $t\geqq 1$ if and only if the partial quotients of the continued fractions for α are bounded. The author proves the following analogous theorem, where αx is replaced by αq^x, q being a fixed integer greater than unity. Theorem: In order that there shall exist a number $C>0$ such that for arbitrary β and $t\geqq 1$ a pair of integers x, y can be found to satisfy

$$0\leqq x<Ct,\quad |\alpha q^x-y-\beta|<\frac{1}{t}$$

it is necessary and sufficient that α shall be of bounded ratio. These last two words are defined as follows. Let δ_n be the nth digit after the decimal point in the "decimal" expansion of α in the scale of q. Then $\lambda=\lambda(n)$ is defined to be the least positive integer with the property that among the first λ n-digited numbers $\delta_{s+1}\delta_{s+2}\cdots\delta_{s+n}$ $(x=0, 1, \cdots, \lambda-1)$ that can be chosen from the decimal expansion of α each of the q^n possible n-digited numbers occurs at least once. It is supposed that the fractional parts of αq^x are everywhere dense over the interval $(0, 1)$. Then α is said to be of bounded ratio if there exists a constant $c_1=c_1(\alpha)$ such that $\lambda(n)/q^n<c_1$ for all $n\geqq 1$. By means of his theory of normal periodic systems the author proves that numbers of bounded ratio do exist and shows how an infinity of them can be constructed. Generalisations of these results to s dimensions are stated without proof. *R. A. Rankin* (Birmingham).

ANALYSIS

Jacobsthal, Ernst. Über das arithmetische und geometrische Mittel. II. Norske Vid. Selsk. Forh., Trondheim **25** (1952), 5–6 (1953).

[For part I see same Forh. **23**, 122 (1951); these Rev. **13**, 117.] An alternative proof of the inequality between the geometric mean and the arithmetic mean, $P^{1/n}\leqq S/n$, of positive numbers $a_1\leqq\cdots\leqq a_n$ is given by induction, as follows: The inequality is trivial if $a_1=a_n$; suppose it has been proved for $a_1=a_{r+1}$, and consider the case $a_1=a_r<a_{r+1}=a_1+(r+1)\epsilon$. Let $a_1'=\cdots=a_{r+1}'=a_1+\epsilon$; otherwise $a_k'=a_k$. Then

$$P\leqq\frac{a_1a_{r+1}P'}{a_1a_{r+1}+\epsilon^2 r}<P'\leqq\left(\frac{S'}{n}\right)^n=\left(\frac{S}{n}\right)^n.$$

E. F. Beckenbach (Los Angeles, Calif.).

Kinokuniya, Yoshio. Mean-position. Mem. Muroran Coll. Tech. **1**, no. 1, 25–27 (1950).

Some graphical observations are made concerning the known result that if $f'(x_0)=\cdots=f^{(n)}(x_0)=0$, $f^{(n+1)}(x_0)\neq 0$, and $f^{(n+1)}(x)$ is continuous, then the value θ in

$$f(x_0+h)-f(x_0)=hf'(x_0+\theta h)$$

satisfies $\lim_{h\to 0}\theta=[1/(n+1)]^{1/n}$. *E. F. Beckenbach.*

Rankin, R. A. A problem concerning the product of the differences of n real variables. Norske Vid. Selsk. Forh., Trondheim **25** (1952), 50–53 (1953).

Let $a_1, a_2, \cdots, a_n$ be n real numbers not all 0. Put

$$D_n=\prod_{i<j}|a_i-a_j|,\quad S_n=\sum_{i=1}^{n}|a_i|.$$

Write $F_n=nD_n^{1/v}/S_n$, $v=\frac{1}{2}n(n-1)$. The author determines $M_n=\sup F_n$ for $2\leqq n\leqq 7$. (In the paper there is a misprint; v is defined in formula (1) as $\frac{1}{2}n(n+1)$.) The author quotes a result of Rogers [Acta Math. **82**, 185–208 (1950); these Rev. **11**, 501] who obtained

$$F_n\leqq\frac{\pi}{2\sqrt{c}}\left(\frac{c^2\pi^2n(n-1)^2}{16}\right)^{1/2n-2}.$$

P. Erdös (Stanford, Calif.).

Mahler, K., and Popken, J. On a maximum problem in arithmetic. Nieuw Arch. Wiskunde (3) **1**, 1–15 (1953). (Dutch. English summary)

Let x be positive, and let n be a positive integer. Denote by V_n the set of all expressions formed from n letters x by $n-1$ additions or multiplications; thus V_3 consists of the four numbers $x+x+x$, $x+x\cdot x$, $x(x+x)$ and $x\cdot x\cdot x$. Let $M_n(x)$ be the maximum element of V_n. It is shown that, if $h=h(n,\nu)$ and $r=r(n,\nu)$ are defined for $1\leqq\nu\leqq n$ by the conditions $n=h\nu+r$, $0\leqq r<\nu$, and if $p_{n,\nu}=h^{\nu-r}(h+1)^r$, then $M_n(x)=\max_{1\leqq\nu\leqq n}(p_{n,\nu}x^\nu)$ and $\lim(M_n(x))^{1/n}=\max_{k\geqq 1}(kx)^{1/k}$.

W. J. LeVeque (Ann Arbor, Mich.).

Tietze, Heinrich. On a gap occurring frequently in mathematical conclusions and a method of partitioning the linear arrangements of the natural numbers into classes. Math. Student **19** (1951), 118–120 (1952).

For an amusing critique of the gap in Steiner's proof of isoperimetry of the circle see F. W. Levi's article in Math. Student **17**, 1–25 (1950) [these Rev. **11**, 680]. This gap is most cogently exposed by Perron's example: The natural numbers, except 1, increase by squaring, yet 1 is not the greatest natural number. Another, though not so perspicuous, example involving series rearrangements is the burden of this paper. *W. Gustin* (Bloomington, Ind.).

Georgiev, G. Formulas of mechanical quadratures with equal coefficients for multiple integrals. Doklady Akad. Nauk SSSR (N.S.) **89**, 389–392 (1953). (Russian)

The author gives formulae of the form

$$\int\int_R P(x,y)dxdy=\lambda\sum_{i=1}^{n}P(x_i,y_i)$$

for double and triple integrals which hold for all polynomials $P(x, y)$ of degree 2 and 3. *G. G. Lorentz.*

Edrei, Albert. On the generation function of a doubly infinite, totally positive sequence. Trans. Amer. Math. Soc. **74**, 367–383 (1953).

Let $\{a_n\}_{-\infty}^{\infty}$ be a doubly infinite sequence and let $f(z)=\sum_{-\infty}^{\infty}a_nz^n$ be its (formal) generating function. The sequence $\{a_n\}_{-\infty}^{\infty}$ is said to be totally positive if for every k and integer sets $m(1)<m(2)<\cdots<m(k)$, $n(1)<n(2)<\cdots<n(k)$ the determinant (*) $|a_{n(i)-m(j)}|_{i,j=1,\cdots,k}$ is non-negative. We shall say that $\{a_n\}_{-\infty}^{\infty}$ is trivial if $a_n=a\rho^n$ $(a>0, \rho>0; -\infty<n<\infty)$. It was shown by I. J. Schoenberg [Courant Anniversary Volume, Interscience, New York, 1948; pp. 351–370; these Rev. **9**, 337] that if

$$f(z)=Cz^k\exp\left[c_1z+c_{-1}z^{-1}\right]\frac{\prod_1^{\infty}(1+\alpha_\nu z)\prod_1^{\infty}(1+\beta_\nu z^{-1})}{\prod_1^{\infty}(1-\gamma_\nu z)\prod_1^{\infty}(1-\delta_\nu z^{-1})}$$

where k is an integer (not necessarily positive), $C\geq 0$, $c_1\geq 0$, $c_{-1}\geq 0$, $\alpha_\nu\geq 0$, $\beta_\nu\geq 0$, $\gamma_\nu\geq 0$, $\delta_\nu\geq 0$, $\sum_1^{\infty}(\alpha_\nu+\beta_\nu+\gamma_\nu+\delta_\nu)<\infty$, and if $\sum_{-\infty}^{\infty}a_nz^n=f(z)$ in some ring $R_1<|z|<R_2$, then $\{a_n\}_{-\infty}^{\infty}$ is totally positive and non-trivial. It was conjectured by Schoenberg that the converse is also true; that is, if $\{a_n\}_{-\infty}^{\infty}$ is totally positive and non-trivial, then $\sum_{-\infty}^{\infty}a_nz^n$ converges in some ring to a function $f(z)$ of the above form. A partial verification of this conjecture, obtained by Aissen, Edrei, Schoenberg, and Whitney, was announced in Proc. Nat. Acad. Sci. U. S. A. **37**, 303–307 (1951) [these Rev. **13**, 17] and published in several articles [Whitney, J. Analyse Math. **2**, 88–92 (1952); Aissen, Schoenberg, and Whitney, ibid. **2**, 93–103 (1952); Edrei, ibid. **2**, 104–109 (1952); these Rev. **14**, 732; see also the paper of Edrei in Canadian J. Math. **5**, 86–94 (1953); these Rev. **14**, 732]. The present paper establishes Schoenberg's conjecture in its complete generality. The demonstration falls roughly into two parts. In the first part it is shown that if $\{a_n\}_{-\infty}^{\infty}$ is totally positive and non-trivial, then

$$\sum_{-\infty}^{\infty}a_nz^n=\exp\psi(z)z^k\frac{\prod_1^{\infty}(1+\alpha_\nu z)\prod_1^{\infty}(1+\beta_\nu z^{-1})}{\prod_1^{\infty}(1-\gamma_\nu z)\prod_1^{\infty}(1-\delta_\nu z^{-1})}$$

where $\alpha_\nu\geq 0, \beta_\nu\geq 0, \gamma_\nu\geq 0, \delta_\nu\geq 0$ and $\sum_1^{\infty}(\alpha_\nu+\beta_\nu+\gamma_\nu+\delta_\nu)<\infty$, and where $\psi(z)$ is single-valued, and analytic except at $z=0$ and $z=\infty$. The arguments used here are direct and turn upon properties of the determinants (*). In the second part it is proved that $\psi(z)=c_{-1}z^{-1}+c_0+c_1z$ where $c_{-1}\geq 0$, $c_1\geq 0$, and c_0 is real. Use is made here of the theory of the distribution of values of meromorphic functions.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Boas, Ralph P. Sur les fonctions possédant une suite de dérivées positives. Bull. Sci. Math. (2) **76**, 142–144 (1952).

L'auteur démontre qu'un corollaire déduit par Dugué [Bull. Sci. Math. (2) **75**, 153–160 (1951); ces Rev. **13**, 330] de son propre théorème II [voir loc. cit.] n'est qu'un cas particulier d'un théorème de S. Bernstein [Leçons sur les propriétés extrémales . . . , Gauthier-Villars, Paris, 1926, p. 197]; le théorème lui-même se déduisant d'ailleurs facilement des résultats de Bernstein. *S. Mandelbrojt.*

Theory of Sets, Theory of Functions of Real Variables

Fodor, G. Über eine mit der verallgemeinerten Kontinuumhypothese äquivalente Behauptung. Publ. Math. Debrecen **2**, 232–234 (1952).

Let $|M|=2^{\aleph_\alpha}$, $M^2=\{(x, y): x\in M, y\in M\}$. Then the following propositions are equivalent. (1) $2^{\aleph_\alpha}=\aleph_{\alpha+1}$. (2) There exists a set $E\subset M^2$ such that, for every $x_0\in M$, $|E\cap\{(x_0, y): y\in M\}|\leq\aleph_\alpha$, and if $A\subseteq M$, where $|A|>\aleph_\alpha$, then there exists at least one $y_0\in A$ such that the set $\{(x, y_0): x\in A\}$ contains $|A|$ elements of E.

F. Bagemihl (Rochester, N. Y.).

Ketskeméty, I. Eine Behauptung, die mit der verallgemeinerten Kontinuumhypothese äquivalent ist. Publ. Math. Debrecen **2**, 235–236 (1952).

Consider the following propositions. (1_α) $2^{\aleph_\alpha}=\aleph_{\alpha+1}$. (2_α) If $|M|=2^{\aleph_\alpha}$, there exists a choice function f, defined for every nonempty finite subset of M, such that every element of M is the value of f for at most $\aleph_\alpha$ of the finite subsets of M. (3) If $n\geq 2$, Euclidean n-dimensional space is the union of n mutually exclusive sets E_i $(1\leq i\leq n)$ such that, for every $(n-1)$-dimensional hyperplane P perpendicular to the ith coordinate axis, $|P\cap E_i|\leq\aleph_0$. The author proves that, for every α, (1_α) and (2_α) are equivalent, and that (2_0) implies (3). Sikorski [Fund. Math. **38**, 18–22 (1951), p. 22; these Rev. **14**, 26] has shown that (1_0) and (3) are equivalent.

F. Bagemihl (Rochester, N. Y.).

Ginsburg, S. Some remarks on a relation between sets and elements. Fund. Math. **39** (1952), 176–178 (1953).

The note is concerned with an error (Theorem II) of Fodor and Ketskeméty [Fund. Math. **37**, 249–250 (1950); these Rev. **13**, 19]. Given a set E, let H be a nonvoid family of sets $h\subseteq E$ and R a relation between any $x\in E$ and h's, so that for each $h\in H$ there is a unique $x\in E$ satisfying $x\in h$, xRh. Denote by G_α the set of all the $x\in E$ such that xRh holds for not less then $\aleph_\alpha$ sets $h\in H$. Theorem 1: If H consists of all finite sets $\subseteq E$ and if $kE>\aleph_\alpha$, then $kG_\alpha=kE$; for $\aleph_\alpha=kE$, G_α may vanish. Theorem 2: If $\aleph_\gamma\leq\aleph_\alpha=kE$ and H consists of all subsets of E each of which has $\aleph_\gamma$ points, then $kG_\alpha=kE$ (kX means the cardinal of X). *G. Kurepa.*

Ginsburg, Seymour. Some remarks on order types and decompositions of sets. Trans. Amer. Math. Soc. **74**, 514–535 (1953).

This paper is a part of the author's thesis and deals with sets $\subseteq R$ (=ordered set of real numbers) and its order types; in particular, the following problem (P) is examined: Given order types σ, $\bar{E}$ such that $\sigma<\bar{E}\leq\lambda$ (=type of R; $\leq$ means $<$ or $=$), does there exist a τ satisfying $\sigma<\tau<\bar{E}$? Each of the following 6 conditions assures τ: 1) $2^{\aleph_0}=k$ E(=cardinal of E) and $A\leq\sigma$ implies $k(E-A)=kE$ (Theorem 1.1); 2) E is the union of $2^{\aleph_0}$ pairwise disjoint sets similar to E (Corollary 1.3; as yet, no type $\neq\lambda$ is known to have that property); 3) $\bar{E}=\alpha\omega$, $\alpha<\alpha 2$ (Theorem 1.3); 4) $E=A\omega$ and A has a fixed point (Theorem 1.4); 5) $\bar{E}=\alpha\omega$, $\alpha=\alpha 2$, $\sigma<\alpha$, $\sigma<\sigma 2$ (Theorem 1.5); 6) $\bar{E}=\eta$ (Theorem 1.6). If E has a fixed point and $X=\{x_0, x_1, \cdots, x_n\}$ is a set of $n+1$ consecutive points in E, there are precisely n distinct types located between $\overline{E-X}$ and E (Theorem 2.2) [this is a counterpart to what happens for dimension types; see Sierpinski, Fund. Math. **14**, 122–126 (1929)]. If $k\sigma=2^{\aleph_0}$ and $\sigma<\lambda$, there is an exact set B dense in R such that $\sigma<\bar{B}<\lambda$ and such that each $x\in B$ is a c-condensation point of B (Theorem 2.4) [this strengthens Sierpinski's result, Atti Accad. Naz. Lincei. Rend. Cl. Sci.

Fis. Mat. Nat. (8) 8, 427–428 (1950); these Rev. 13, 19]. If $kE=2^{\aleph_0}$, E is decomposable into $2^{\aleph_0}$ disjoint exact sets of property (A) (Theorem 3.1). If E is similar to $R-E$, then E has no fixed point and is not exact (Theorem 3.2). Theorem 4.2 assures the existence of an E of power $2^{\aleph_0}$ having no fixed point, although for each similarity f of E into E there is a point x_f such that $f(x_f)=x_f$. Terminology: x_0 is a fixed point of E if for each similarity f: $E\to E$ one has $f(x_0)=x_0$; c-condensation point of E is each $x \in R$ such that each neighbourhood of x contains $2^{\aleph_0}$ points of E preceding x and as many $>x$; $E \in$ (A), if $kE=2^{\aleph_0}$ and contains no two similar sets of that power. [It is to be remarked that Lemma 1.6 attributed to Shepherdson [Proc. London Math. Soc. (3) 1, 291–307 (1951); these Rev. 13, 330] and Theorems 1.6 and 1.7 (where $\equiv$ is to be read $=$) are contained in a note of the reviewer [Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 3, 145–151 (1948); these Rev. 10, 437].] *G. Kurepa* (Zagreb).

Aronszajn, N. Characterization of types of order satisfying $\alpha_0+\alpha_1=\alpha_1+\alpha_0$. Fund. Math. 39 (1952), 65–96 (1953).

The principal result states that $\alpha_0+\alpha_1=\alpha_1+\alpha_0$ if and only if either $\alpha_0=\alpha_1\omega+\delta+\alpha_1\omega^*$ or $\alpha_1=\alpha_0\omega+\delta+\alpha_0\omega^*$ with some type of order δ, or else

$$\alpha_0=\gamma_0+\sum_{0<\xi<\xi_0}\gamma(\xi)+\gamma_1,\quad \alpha_1=\gamma_0+\sum_{\xi_0<\xi<1}\gamma(\xi)+\gamma_1,$$

where ξ_0 is real, $0<\xi_0<1$ and γ_0, γ_1, and $\gamma(\xi)$ are types of order, $\gamma(\xi)$ being defined in the interval $0<\xi<1$ and satisfying the conditions: (i) $\gamma(\xi_1)=\gamma(\xi_2)$ if $\xi_2-\xi_1=m+m_0\xi_0$ for some integers m and m_0; (ii) $\gamma(\xi)=\gamma_1+\gamma_0$ for all $\xi=m+m_0\xi_0$ with any integers m and m_0. *B. Jónsson.*

Edelstein, Michael. On contact between sets of points. Riveon Lematematika 6, 45–52 (1953). (Hebrew)

The following theorem is proved: Let U_1, U_2, T be bounded regions in the plane; let d be a supporting line of the union $\bar{U}_1+\bar{U}_2$ such that the intersections $d\bar{U}_1$ and $d\bar{U}_2$ are non-empty and are contained in disjoint half-lines of d; and let $V^*(T, \bar{U}_1+\bar{U}_2)$ be connected; then there are at least two vectors such that the translate of T determined by each of them is in contact with both U_1 and U_2. Here $V^*(T, U)$ denotes the set of all points x such that the translation of T by the vector $\overrightarrow{ax}$ meets U [Edelstein, same Riveon 1, 91–94 (1947); these Rev. 9, 137]. Two connected, open sets are said to be in contact if they are disjoint and their closures intersect. *M. Jerison* (Lafayette, Ind.).

Straszewicz, S. Sur les directions singulières par rapport à un ensemble de points. Fund. Math. 39 (1952), 128–130 (1953).

Let E be a subset of three-dimensional euclidean space. A given direction is said to be singular with respect to the set E if and only if there exists no line having this direction which intersects E in precisely one point. This paper makes use of simple properties of closed and bounded convex sets to obtain a proof of the following theorem proposed by Sierpinski [Matematiche, Catania 6, 132–134 (1951); these Rev. 13, 673]: The set of singular directions, with respect to a closed and bounded set of points, is at most denumerable. The author remarks that the theorem extends without difficulty to n-dimensional euclidean space, where n exceeds 3. *D. W. Hall* (College Park, Md.).

Stanojević, Časlav V. On a system of the set equations. Bull. Soc. Math. Phys. Serbie 4, no. 3–4, 39–41 (1952). (Serbo-Croatian. English summary)

The simultaneous Boolean equations

$$(X\cap B')\cup(Y'\cap A)=A,\quad (X'\cap B)\cup(Y\cap A')=B$$

are solved. *E. Hewitt* (Seattle, Wash.).

Germansky, Baruch. On the Fekete-systems of sets consisting of an arbitrary finite number of finite and closed intervals of a straight line. I. Riveon Lematematika 6, 61–78 (1953).

Let M be an arbitrary closed set in the plane. For each m we consider m points p_k varying arbitrarily on M, and choose a system of points for which the product $\prod|p_ip_k|$, $i<k$; $i, k=1, 2, \cdots, m$, is a relative maximum; here p_ip_k denotes the distance of p_i and p_k. Systems of this kind have been introduced and investigated by M. Fekete [Math. Z. 17, 228–249 (1923)]. The author of the present paper deals with the determination of such systems in the interesting case of a set M consisting of a finite number of disjoint intervals on the real axis. It is shown that if m is sufficiently large, these systems will necessarily contain the end-points of all intervals. Thus they are special cases of the classical systems considered by Heine and Stieltjes for which differential equations can be established. The first proof of one of the main results is due to Dvoretzky. *G. Szegö.*

Buck, R. Creighton. Generalized asymptotic density. Amer. J. Math. 75, 335–346 (1953).

Soit X un ensemble, (K_n) une suite croissante de parties de X formant un recouvrement de X; une partie de X est dite bornée si elle est contenue dans un K_n. On considère une suite (μ_n) de mesures positives sur X, X et les K_n étant mesurables pour toutes ces mesures. On suppose en outre que $\mu_n(X)=1$ et que pour tout j, $\mu_n(K_j)$ tend vers 0 avec $1/n$. Pour tout ensemble S mesurable pour toutes les μ_n, l'auteur définit alors la densité $D(S)$ de S comme la limite de $\mu_n(S)$ lorsque n tend vers $+\infty$; les densités supérieure et inférieure $\bar{D}(S)$ et $\underline{D}(S)$ se définissent de même. Ces notions généralisent de façon évidente les notions usuelles de densité, correspondant au cas où X est l'ensemble des entiers naturels et μ_n définie par la masse $1/n$ aux points $1, 2, \cdots, n$. Supposant que chacune des μ_n a un support "borné", l'auteur démontre le théorème suivant: en notant $A\dot{\subset}B$ la relation "$A-B$ est borné", si (A_n) est une suite telle que $A_n\dot{\subset}A_{n+1}$ et si $\Delta=\lim \bar{D}(A_n)$, $\delta=\lim \underline{D}(A_n)$, alors il existe une A tel que $\bar{D}(A)=\Delta$, $\underline{D}(A)=\delta$ et $A_n\dot{\subset}A$ pour tout n. La suite des μ_n lui permet ensuite de définir une notion de (μ)-sommabilité d'une fonction f définie dans X; on dit que $(\mu)\text{-}\lim f(x)=L$ si $\lim_{n\to\infty}\int f d\mu_n=L$. L'auteur étudie le rapport de cette notion avec celle de la convergence "en densité" ($f(x)$ tend "en densité" vers L s'il existe un ensemble A de densité 0 tel que, sur $X-A$, $f(x)$ tende vers L suivant la base de filtre des complémentaires des K_n). Il applique en particulier ses résultats au cas classique rappelé ci-dessus, ce qui lui donne des théorèmes sur la convergence et la sommabilité $(C, 1)$ des séries à termes constants ou des séries de fonctions orthogonales. *J. Dieudonné.*

Hewitt, Edwin. A note on measures in Boolean algebras. Duke Math. J. 20, 253–256 (1953).

Since infinite operations in Boolean algebras are not necessarily preserved by Boolean isomorphisms, it is possible that a finitely (but not necessarily countably) additive measure on a Boolean algebra be carried by an isomorphism

to a countably additive measure on a class of sets. The author's principal result is that if this happens for every isomorphic representation by a class of sets, then the original measure must be countably additive. *P. R. Halmos.*

Jessen, Børge. On strong differentiation. Mat. Tidsskr. B. 1952, 90–91 (1952).

The author constructs a Borel measurable set C in the torus space of infinitely many dimensions so that the characteristic function of C is not equal almost everywhere to the strong derivative of its indefinite integral. Thus he replaces his previous construction [Mat. Tidsskr. B. 1950, 54–57; these Rev. 12, 324] for which the proof of the above property is deficient. *H. M. Schaerf* (Princeton, N. J.).

Četković, Simon. Sur la différentiabilité des deux familles des fonctions réelles. Bull. Soc. Math. Phys. Serbie 4, no. 3–4, 53–57 (1952). (Serbo-Croatian. French summary)

Let A and B be complementary sets of natural numbers, each containing an infinite number of primes. Let S be the set of real numbers which can be written in the form p/q, where p is an integer, $q \in B$, and p and q are relatively prime. Let $F(x)=0$ on S' and let $F(p/q)=q^{-\alpha}$ for $p/q \in S$, α being an arbitrary positive real number. Then, if $0<\alpha\leqq 1$, F is nowhere differentiable but is continuous on an everywhere dense set; if $1<\alpha$, then F is discontinuous on a dense set but is also differentiable on a dense set. *E. Hewitt.*

Wilansky, Albert. Two examples in real variables. Amer. Math. Monthly 60, 317 (1953).

Examples are as follows. 1. A bounded Lebesgue-integrable function f such that there is no Riemann-integrable function g with $f=g$ almost everywhere. 2. A Riemann-integrable function which is not of Baire class I.

Extract from paper.

Goffman, Casper. Definition of the Lebesgue integral. Amer. Math. Monthly 60, 251–252 (1953).

An integral equivalent to the Lebesgue integral is defined as the limit of a sequence of integrals of continuous functions. The definition is essentially that given by L. Tonelli [Ann. Mat. Pura Appl. (4) 1, 105–145 (1924)].

U. S. Haslam-Jones (Oxford).

*Goffman, Casper. **Real functions.** Rinehart & Co., Inc., New York, N. Y., 1953. xii+263 pp. $6.00.

Table of contents: 1. Sets and operations. 2. Equivalence of sets. Cardinal numbers. 3. The real numbers. 4. Limit theorems. 5. Simple properties of sets. 6. The Cantor ternary set. 7. Functions. 8. Sequences of functions. 9. The derivative of a function. 10. Order types and ordinals. 11. Borel sets and Baire functions. 12. Applications of well-ordering. 13. Measure and measurable sets. 14. Metric properties of sets. 15. Measurable functions. 16. Approximation of measurable functions. 17. The Lebesgue integral and the Riemann integral. 18. The Lebesgue integral as a set function. 19. The fundamental theorem of the calculus. 20. Planar measure and double integration. This book is the outgrowth of a course given by the author to first-year graduate students and is written as a textbook in the style of the classical volumes of F. Hausdorff, H. Hahn, and C. Carathéodory. Here are some of its features. The operations of union and intersection of sets are denoted by the lattice-theoretical symbols $\cup$ and $\cap$. Inf and sup are used for g.l.b. and l.u.b. Stone's sum and product of sets and the notion of Boolean ring are mentioned. A subset S of $U=[0,1]$ is shown to be Lebesgue measurable if and only if, for every $\epsilon>0$, there is a positive integer $n=n(\epsilon)$ and there are disjoint intervals (open or closed) $I_1, I_2, \cdots, I_n$ such that $S=(\bigcup_{k=1}^{n} I_k \cup E_2)-E_1$ where E_1 and E_2 have exterior measures less than ϵ. [Remark by the reviewer: It would have been simpler to estimate the approximation of S by $S^*=\bigcup I_k$ as exterior measure of Stone's difference $S-S^*$.] The Lebesgue integral of a function $f(x)$, $x \in U$, is first defined for measurable simple functions and then extended to arbitrary measurable functions by means of "vertical" partitions, called "nets". The upper and lower Riemann integrals are introduced as usual and a corresponding "horizontal" definition for the L-integrals of a bounded function given, clearly showing that the superiority of Lebesgue's integral over Riemann's is not due to the use of vertical partitions but of a finer measure on U, yielding a finer system of horizontal partitions. Two defects of the Riemann integral are pointed out which subsequently are shown to be remedied by Lebesgue's theory. Each chapter of the book is followed by many exercises of various degrees of difficulty, numbered according to the section to which they refer. Delicate topics (e.g., nonuniform convergence of sequences of functions) are carefully and smoothly introduced. The presentation is inviting and fairly free of misprints. *Chr. Pauc* (Nantes).

Stöhr, Alfred. Bemerkungen über Kettenbruchintegrale. Math. Nachr. 8, 157–165 (1952).

The author mentions first the priority question for the ideas involved in obtaining a continued fraction integral, and acknowledges the priority of Puig Adam [Revista Mat. Hisp.-Amer. (4) 11, 180–190 (1951); these Rev. 13, 540]. He continues the discussion of his earlier paper on continued fraction integrals [Math. Nachr. 6, 103–107 (1951); these Rev. 14, 24] by noting that, with the introduction of Riemann-Stieltjes integration in place of Riemann integration in the definition, ordinary continued fractions are included as special cases of continued fraction integrals. The application of the continued fraction integral with matrix coefficients to the $n\times n$ matrix differential system $Y_1'=P_{11}Y_1+P_{12}Y_2$, $Y_2'=P_{21}Y_1+P_{22}Y_2$, and the related $n\times n$ matrix Riccati equation

$$Z'=-ZP_{21}Z+P_{11}Z-ZP_{22}+P_{21},$$

where $ZY_2=Y_1$, is indicated. *W. T. Scott.*

Nosarzewska, M. On uniform convergence in some classes of functions. Fund. Math. 39 (1952), 38–52 (1953).

The author first recalls the known result that if a sequence of monotonic functions, defined in an open interval, converges to a continuous function f, then the sequence converges almost uniformly to f; that is, the sequence converges uniformly to f on any closed and bounded subinterval. It is shown that the result remains valid if the sequence of monotonic functions is replaced by a sequence of convex functions, or by a sequence of generalized convex functions, or by a sequence of functions with pth difference having a constant sign. The implication of almost uniform convergence still holds for all these classes of functions when the assumption of convergence is replaced by the weaker assumption of asymptotic convergence, or convergence on a dense subset, or even pointwise approximation. But the implication does not necessarily hold for a convergent sequence of subharmonic functions. *E. F. Beckenbach.*

Mulè, Giovanni. Su un criterio di convergenza uniforme. Boll. Un. Mat. Ital. (3) **8**, 43–44 (1953).

Let $f, f_1, f_2, \cdots$ be functions on a subset I of a Hausdorff space to a metric space. Then the sequence $\{f_n\}$ converges uniformly on I to f if (and only if) each subsequence of $\{f_n\}$ has a subsequence convergent uniformly on I to f. The sufficiency of a criterion for uniform convergence due to Picone [same Boll. (3) **7**, 106–108 (1952); these Rev. **14**, 148] is an immediate corollary. *T. A. Botts.*

Cafiero, Federico. Sulle funzioni misurabili rispetto ad una ed assolutamente continue rispetto ad un'altra variabile. Ricerche Mat. **1**, 227–240 (1952).

Let S_1 be the class of functions $f(x, y)$ defined in the rectangle $R=I_1\times I_2$ which are measurable in x for almost all y and absolutely continuous in y for almost all x. Then each function $f \in S_1$ possesses the following property: (α) given $\epsilon>0$, $\omega>0$, there is a set $e_\epsilon\subset I_1$, $|e_\epsilon|<\epsilon$ and a $\delta>0$ such that $\sum|f(x, d_i)-f(x, c_i)|<\omega$ for any $x \in I_1-e_\epsilon$ and any system of nonoverlapping intervals (c_i, d_i), $\sum|d_i-c_i|<\delta$. Let $\{f_n\}\subset S_1$ be such that (i) property (α) is satisfied uniformly by f_n, (ii) for any $\epsilon>0$ there is a set $e_\epsilon^{(n)}\subset I_1$, $|e_\epsilon^{(n)}|<\epsilon$ such that $|f_n(x, y)|<M$ in $R-e_\epsilon^{(n)}\times I_2$, (iii) for any y of a dense subset $H\subset I_2$ there is a subsequence $\{f_n'\}\subset\{f_n\}$ which converges a.e. in I_1; then there is a subsequence $\{f_n'\}\subset\{f_n\}$ such that given $\epsilon>0$ there is a set $I_\epsilon\subset I_1$, $|I_\epsilon|<\epsilon$ such that $\{f_n'\}$ is uniformly convergent in $I_2\times(I_1-I_\epsilon)$ to a function $f \in S_1$. Finally, property (α) is satisfied uniformly if $\int\int|\partial f_n/\partial y|^{1+\alpha}dxdy\leqq A$ for $\alpha>0$. *M. Cotlar* (Buenos Aires).

Giuliano, Landolino. Una proprietà delle successioni di funzioni generalmente a variazione limitata. Ann. Scuola Norm. Super. Pisa (3) **6**, 99–107 (1952).

A sequence of functions $\{f_n(x, y)\}$ defined in a square Q is said to be equally almost lipschitzian if given $\epsilon>0$ there is a number $M>0$ and a sequence of subsets $e_n\subset Q$ such that $|Q-e_n|<\epsilon$ and

$$|f_n(x', y')-f_n(x'', y'')|\leqq M\{(x'-x'')^2+(y'-y'')^2\}^{1/2}$$

for $(x', y') \in e_n$, $(x'', y'') \in e_n$. The notion of an equally almost bounded sequence is defined similarly. Theorem. Let $f_n(x, y)$ be a sequence of measurable functions such that (i) $\{f_n\}$ is equally almost bounded in Q,

$$\text{(ii)}\qquad \int_0^1 V_y^{(n)}(x, 1)dx+\int_0^1 V_x^{(n)}(1, y)dy<K,$$

where $V_y^{(n)}(x, 1)$ is the total variation in $(0, 1)$ of $f_n(x, y)$ considered as a function of y; then the functions $f_n(x, y)$ are equally almost lipschitzian. *M. Cotlar.*

Taylor, S. J. On the volume of a topological cube. J. London Math. Soc. **28**, 214–220 (1953).

The topological image Q (in Euclidean three-space) of a cube q is termed a topological cube. The images of the vertices, edges, and faces of q (under a fixed homeomorphism from q onto Q) are termed the vertices, edges, and faces of Q. The distance $d(P_1, P_2)$ of two points P_1, P_2 of Q is defined as the greatest lower bound of the lengths of the paths connecting P_1 and P_2 in Q. The distance $d(E_1, E_2)$ of two compact subsets of Q is then defined in an obvious manner. Let $S_1, S_1', S_2, S_2', S_3, S_3'$ be the three pairs of opposite faces of Q, and let a, b, c denote the distances (in the sense indicated above) $d(S_1, S_1')$, $d(S_2, S_2')$, $d(S_3, S_3')$. Finally, let V denote the three-dimensional Lebesgue measure of Q. Generalizing an earlier result of Besicovitch relating to topological squares, the author shows that $V\geqq abc$. The proof is somewhat involved, even though quite direct and elementary. *T. Radó* (Columbus, Ohio).

Lesovoĭ, B. V. A measure of area in a two-parameter family of curves on a surface. Trudy Sem. Vektor. Tenzor. Analizu **6**, 447–493 (1948). (Russian)

The paper contains contributions to the following problem. A two-parameter system of curves is given on a surface F. When is it possible to introduce a measure σ for the curves such that (a) if S_1 and S_2 are sets of curves without a common element, then $\sigma(S_1)=\sigma(S_2)$, (b) if under an (intrinsically) isometric mapping of F on F^* the set of curves S goes into S^*, then $\sigma(S)=\sigma(S^*)$?

The curves are considered as sets of line elements and measure is defined in terms of two distances: the distance of the points carrying the line elements and the deviation of the directions of the line elements. The resulting formalism is much too involved to be outlined here. The problem is not solved in general, but for certain special distances. For instance, in geometric measure the point distance is the length of the arc A of the curve of the given system Σ through the two points and the deviation of the directions is the angle between one direction and the direction obtained from the other by parallel transport along A. It is shown that a geometric measure exists for a curve-system Σ if and only if the curves of Σ through a given point p have at p all the same geodesic curvature.

Measures invariant under conformal mapping are also discussed and they are connected with geometric measure by the following theorem. For a given curve-system Σ on a given surface F there exists an intrinsically unique surface F^* on which F can be mapped conformally such that Σ goes into a system $\bar{\Sigma}$ for which the geodesic curvatures behave as above. *H. Busemann* (Los Angeles, Calif.).

Cecconi, Jaurès. Confronto fra recenti definizioni di variazione totale per trasformazioni piane. Boll. Un. Mat. Ital. (3) **8**, 10–19 (1953).

Let T denote the transformation defined by the continuous functions $x=x(u, v)$, $y=y(u, v)$, $0\leqq y\leqq 1$, $0\leqq v\leqq 1$. The concept of bounded variation, for such transformations T, has been defined in various ways by the reviewer, by Reichelderfer, and by Cesari [for details, see, for example, the reviewer's book "Length and area", Amer. Math. Soc. Colloq. Publ., vol. 30, New York, 1948; these Rev. **9**, 505]. While the definitions of bounded variation just referred to are known to be equivalent, the author shows that a recent definition proposed and utilized by H. Okamura [Mem. Coll. Sci. Kyoto. Ser. A. Math. **26**, 5–14 (1950); these Rev. **12**, 488, 1003] yields a more restricted class of transformations. The explicit definitions, all based on the topological index, are too involved for re-statement here. *T. Radó.*

Seki, Setsuya. On the change of variables in the multiple integrals. J. Math. Soc. Japan **4**, 218–230 (1952).

Let D be a bounded domain in Euclidean n-space E^n, and let $f: D\to E^n$ denote a continuous mapping from D into E^n. Then f is termed an A-function on D if it satisfies the following conditions. (A_1) f is a homeomorphism. (A_2) If S is a set of Lebesgue measure zero in D, then its image fS is also of Lebesgue measure zero. (A_3) f is totally differentiable almost everywhere in D. The author proves that if f is an A-function in D, then the standard transformation formula for n-tuple integrals holds for the transformation defined by f. A similar result is established for the case when

condition (A_1) is replaced by the less restrictive requirement that f is locally homeomorphic almost everywhere in D. These results are special cases of general theorems announced previously by the reviewer and Reichelderfer [Proc. Nat. Acad. Sci. U. S. A. 35, 678–681 (1949); these Rev. 11, 588]. *T. Radó* (Columbus, Ohio).

Theory of Functions of Complex Variables

*Ahlfors, Lars V. **Complex analysis. An introduction to the theory of analytic functions of one complex variable.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. xii+247 pp. $5.00.

L'ouvrage est un exposé destiné aux étudiants de première année, de la théorie des fonctions analytiques d'une variable complexe. Le programme est classique; le livre contient toutefois quelques paragraphes très remarquables qui peuvent servir d'introduction à des études plus poussées et concernent le problème de Dirichlet, les fonctions sousharmoniques, les surfaces de Riemann, et quelques autres points. Les dons de style de l'auteur lui permettent d'évoquer en très peu de mots des images qui préparent le lecteur à l'intuition et à la comprehension des démonstrations rigoureuses; ces qualités contribuent à faire de l'ouvrage une réussite très remarquable sur de nombreux points.

Le chapitre I donne les éléments: opérations sur les nombres complexes, transformation homographique; plan complexe obtenu comme projection stéréographique de la sphère. Les fonctions analytiques $f(z)$ sont définies au chapitre II par l'existence de la dérivée. L'usage fait des variables z, $\bar{z}$ est heureux, mais l'écriture de la condition $\partial f/\partial\bar{z}=0$ eût gagné en clarté à être reportée au chapitre III après la série de Taylor; il en est de même de l'égalité $f(z)=2u(z/2, z/2i)-f(0)$, qui exprime $f(z)$ à partir de sa partie réelle u.

Le chapitre III étudie l'intégrale $\int_\gamma f(z)dz$ (on suppose toujours $f(z)$ analytique dans un domaine D contenant γ); la signification géométrique de l'indice

$$n(\gamma, a)=(2i\pi)^{-1}\int_\gamma (z-a)^{-1}dz$$

qui figure dans le théorème de Cauchy,

$$(2i\pi)^{-1}\int f(z)(z-a)^{-1}dz=n(\gamma, a)f(a),$$

est soulignée très heureusement. On regrettera par contre que le caractère simplement connexe d'un domaine D soit défini comme la propriété du complémentaire $C(D)$ d'être connexe: c'est là user et abuser de propriétés topologiques du plan dont il serait opportun, même à ce niveau d'enseignement, d'indiquer le caractère particulier. La notion de cycle homologue à zero dans D, très intuitive, est exposé à la fin du chapitre, en vue d'énoncer avec précision le théorème de Cauchy.

Le chapitre IV donne les propriétés des suites de fonctions analytiques, des produits infinis, y compris le théorème de Weierstrass; une place est réservée aux fonctions particulières et à la démonstration de la formule de Stirling par la méthodes des résidus. Un exposé, très bref, qui aurait gagné à être illustré de quelques applications simples, est donné de la théorie des familles normales, ainsi qu'une démonstration du théorème rélatif à la représentation conforme d'un domaine simplement connexe sur un disque.

Le chapitre V traite des fonctions harmoniques et du problème de Dirichlet. La différentielle

$$^*du=-(\partial u/\partial y)dx+(\partial u/\partial x)dy,$$

conjuguée de du et la relation $\int_\gamma(u_1\,{}^*du_2-u_2\,{}^*du_1)=0$ pour tout cycle $\gamma\sim 0$ sont utilisées. La formule de Poisson et ses nombreuses applications sont exposées succinctement; un paragraphe introduit les fonctions sousharmoniques continues et expose la solution de Perron du problème de Dirichlet; ce chapitre un peu chargé se termine par l'étude de quelques représentations conformes de domaines multiplement connexes. Le chapitre VI, plus court, applique les résultats précédents à l'étude des fonctions algébriques et des solutions des équations différentielles linéaires, de la série hypergéométrique en particulier. L'ensemble constitue ainsi un exposé déjà très complet et cependant remarquablement accessible de la théorie des fonctions analytiques d'une variable complexe. *P. Lelong* (Lille).

*Thron, Wolfgang J. **Introduction to the theory of functions of a complex variable.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1953. x+230 pp. $6.50.

Dans cet ouvrage, destiné aux étudiants, et qui ne contient aucun réesultat nouveau, l'auteur expose la théorie classique des fonctions analytiques.

Il introduit donc d'abord (sections 1 à 14) les définitions et les résultats utilisés dans cette théorie, à savoir: éléments de topologie générale, notion de nombre, de fonction continue et différentiable, courbes planes et notion de topologie combinatoire (la démonstration du théorème de Jordan est donnée).

La théorie proprement dite des fonctions analytiques commence par l'intégration complexe et le théorème de Cauchy $\int_C f(z)dz=0$ (qui n'est donné que dans le cas où f est holomorphe sur C). Vient ensuite une étude des suites de fonctions et des séries infinies, suivie de considérations générales sur les espaces fonctionnels. Enfin sont donnés les principes essentiels de la théorie (prolongement analytique, principe de symétrie, représentation conforme, fonctions elliptiques et modulaires, surfaces de Riemann) mais un peu succinctement.

On peut reprocher à ce livre de ne pas mettre en valeur les points importants et d'introduire des notions, comme celle du transfini, qui ne sont pas essentielles pour la théorie. Le souci de ne rien supposer connu et de ne renvoyer à aucun autre ouvrage a amené l'auteur à accorder trop de place aux préliminaires; enfin le point de vue assez abstrait adopté (par exemple, sur les surfaces de Riemann) laisse peu de place aux applications qui familiariseraient l'étudiant avec les notions acquises. Mais tel qu'il est, cet ouvrage peut rendre des services à ceux qui veulent enseigner, avec rigueur, la théorie des fonctions d'une variable complexe. *J. Lelong-Ferrand* (Lille).

Petracca, Antonio, and Levi, Beppo. Complement to the note: Study of a polydromic function. Math. Notae 12–13, 48–49 (1952). (Spanish)

For the note referred to in the title see Math. Notae 11, 124–138 (1951); these Rev. 14, 546. *A. J. Macintyre.*

Duparc, H. J. A., Lekkerkerker, C. G., and Peremans, W. An elementary proof of a formula of Jensen. Math. Centrum Amsterdam. Rapport **ZW 1952-021**, 4 pp. (1952).

The formula in question is

$$\sum_{s=0}^{\infty} e^{-(a+ss)}\frac{(a+ss)^s}{s!}=\frac{1}{1-s} \quad \text{for} \quad |z|<1, \quad |ze^{-s}|<\frac{1}{e}.$$

Mergelyan, S. N. On the representation of functions by series of polynomials on closed sets. Amer. Math. Soc. Translation no. **85**, 8 pp. (1953).

Translated from Doklady Akad. Nauk SSSR (N.S.) **78**, 405–408 (1951); these Rev. **13**, 23.

Mergelyan, S. N. On a theorem of L. A. Lavrent'ev. Amer. Math. Soc. Translation no. **86**, 7 pp. (1953).

Translated from Doklady Akad. Nauk SSSR (N.S.) **77**, 565–568 (1951); these Rev. **12**, 814. The title has been misprinted; read "M. A. Lavrent'ev" for "L. A. Lavrent'ev".

Elliott, H. Margaret. On approximation to analytic functions by rational functions. Proc. Amer. Math. Soc. **4**, 161–167 (1953).

Let C designate a finite number of mutually exterior analytic Jordan curves. Let there be given a system of points α_{nj}, $j=1, \cdots, n$; $n=1, 2, \cdots$, which lie exterior to and away from C. Then if $f(z)$ is analytic in C and if $f^{(k)}(z)$ exists and is continuous on C with modulus of continuity $\omega(\delta)$, the author shows that rational functions

$$r_n(z)=(a_{n0}z^n+\cdots+a_{nn})/(z-\alpha_{n1})(z-\alpha_{n2})\cdots(z-\alpha_{nn})$$

can be found such that $|f(z)-r_n(z)|\leqq M\omega(1/n)/n^k$, $z \in C$. The inverse question is considered, and a theorem is derived which relates the degree of approximation by rational functions $r_n(z)$ with the modulus of continuity of f on C.

P. Davis (Washington, D. C.).

Boas, R. P., Jr. Integral functions with negative zeros. Canadian J. Math. **5**, 179–184 (1953).

Es ist ein klassisches Resultat, dass für ganze Funktionen $f(z)$ von positiver Ordnung $\rho<1$ mit lauter negativen Nullstellen und $f(z)=1$ die beiden Bedingungen $(r\to\infty, A>0)$ $\log f(r)\sim A\pi r^\rho\cdot\operatorname{cosec}\pi\rho$ und $n(r)\sim A\cdot r^\rho$ aequivalent sind. Verf. beweist, dass auch die Bedingung

$$\int_0^r x^{-1-\sigma}\{\log|f(-x)|-\pi n(x)\cdot\operatorname{cotg}\pi\sigma\}dx \sim\pi A(\rho-\sigma)^{-1}(\operatorname{cotg}\pi\rho-\operatorname{cotg}\pi\sigma)r^{\rho-\sigma}$$

für irgend ein positives $\sigma<1$ mit den vorangehenden aequivalent ist. Dies verallgemeinert ein Resultat von Paley und Wiener in Falle $\rho=1$. *A. Pfluger* (Zürich).

Siddiqi, Jamil Ahmad. Quelques théorèmes d'unicité. C. R. Acad. Sci. Paris **236**, 1727–1729 (1953).

The author generalizes the theorem that a function of exponential type in a half-plane must vanish identically if it is of exponential decrease on the boundary. In his theorem the function is studied in a curvilinear angle which is nearly a half-plane and the condition of exponential decrease is replaced by one connecting the growth allowed inside the angle and the closeness of the angle to a half-plane. As applications the author states two theorems in which a function of prescribed growth in $x>0$, with zeros at sufficiently many real points, must vanish identically. The hypotheses are more general than those previously used in investigations of this kind, but are too complex to reproduce here. *R. P. Boas, Jr.* (Evanston, Ill.).

Noble, M. E. Non-measurable interpolation sets. III. A theorem of B. J. Maitland. Quart. J. Math., Oxford Ser. (2) **4**, 11–18 (1953).

[For part II see Proc. Cambridge Philos. Soc. **47**, 733–740 (1951); these Rev. **13**, 336.] The theorem of Maitland states that for an entire function of order ρ and type κ, and for sufficiently small d, there exists $\{\zeta_n\}$ with $|\zeta_n|\to\infty$ and $\lim|z|^{-\rho}|f(z)|=\kappa$ as $z\to\infty$ in circles of center ζ_n and radius $d|\zeta_n|^{1-\frac{1}{2}\rho}$. The author refines this to obtain a lower bound for the upper density of the set of r's such that $\log|f(z)|\geqq\mu r^\rho$ in $|z-\zeta(r)|\leqq dr^{1-\frac{1}{2}\rho}$, $|\zeta(r)|=r$, $\mu<\kappa$. The methods are those of paper I of this series [ibid. **47**, 713–732 (1951); these Rev. **13**, 336]. *R. P. Boas, Jr.* (Evanston, Ill.).

Naftalevič, A. G. On interpolation of functions meromorphic in the unit circle. Doklady Akad. Nauk SSSR (N.S.) **88**, 205–208 (1953). (Russian)

Seven theorems relating the order of a function $f(z)$ meromorphic in $|z|<1$ to the leading terms

$$\sum_{\mu=-l_n}^{m_n} a(n,\mu)(z-\lambda_n)^\mu$$

in the Laurent expansions of $f(z)$ near a sequence of poles λ_n are stated without proof together with supplementary remarks. The results are in general analogous to those of Whittaker [Proc. London Math. Soc. (2) **40**, 255–272 (1935)] for functions meromorphic in the whole plane. However, the possibility of functions of bounded type and special hypotheses about the manner in which the λ_n tend to $|z|=1$ involve additional complications. The problem is also extended by allowing or excluding the possibility of additional poles, and the cases $m_n>0$, $m_n<0$, m_n and l_n bounded, $-m_n=l_n=1$ are all given separate treatment.

A. J. Macintyre (Aberdeen).

Erwe, Friedhelm. Über die Lücken bei Laurentreihen. Arch. Math. **4**, 28–30 (1953).

Let f be meromorphic, and have around the origin the expansion $f(z)=a_0z^{n_0}+a_1z^{n_1}+\cdots$ where $n_0<n_1<n_2<\cdots$, and no a_ν zero. The author wishes to determine the exponents n_ν (not necessarily positive). He first observes that $n_0=\operatorname{Res}(f'/f)=\lim_{z\to0}zf'(z)/f(z)=\omega(f)$. Let D be the differential operator zd/dz, and define functions α_k by

$$\alpha_k=\det\begin{bmatrix} f & Df & \cdots & D^kf \\ Df & D^2f & \cdots & D^{k+1}f \\ \cdot & \cdot & \cdots & \cdot \\ D^kf & \cdot & \cdots & D^{2k}f \end{bmatrix}.$$

Then $\omega(\alpha_\nu)=n_0+n_1+\cdots+n_\nu$ for $\nu=0, 1, 2, \cdots$.

R. C. Buck (Madison, Wis.).

Lehto, Olli. Sur la théorie des fonctions méromorphes à caractéristique bornée. C. R. Acad. Sci. Paris **236**, 1943–1945 (1953).

If $f(z)$ is a meromorphic function in $|z|<1$ with bounded characteristic, almost all of whose boundary values $f(e^{i\varphi})$ lie in a closed set Γ with connected complement D, then either $f(z)$ assumes no value of D or else assumes all values in D outside a set of zero capacity. *P. R. Garabedian.*

Hiong, King-Lai. Sur les fonctions holomorphes dans le cercle unité ne prenant une valeur que p fois et admettant une valeur exceptionnelle au sens de Picard-Borel ou au sens de R. Nevanlinna. C. R. Acad. Sci. Paris **236**, 1628–1630 (1953).

In this note the author continues his work [same C. R. **236**, 1322–1324 (1953); these Rev. **14**, 741] on functions $f(z)=c_0+c_kz^k+\cdots$ $(c_0\neq 0, 1;\ c_k\neq 0)$ regular in $|z|=r<1$ which have one exceptional value. The following result is announced. If $f(z)$ has 1 as exceptional value in the sense of Picard-Borel or as defective value in the sense of Nevanlinna, and if $f(z)$ has at most p zeros, then at the points of the unit circle outside circles of radius δ about 0, and of radii δ_ν about the zeros a_ν,

$$\log|f(z)|<(1-r)^{-3}\{H\Omega(c_0,\delta,\delta_\nu)+K\log 2/(1-r)\},$$

H and K absolute constants. Consequences of this result indicated by the author include normality of families and an inequality for $|f'(0)|$ of the Landau type.

J. Korevaar (Madison, Wis.).

Hiong, King-Lai. Sur la théorie des défauts relative aux fonctions holomorphes dans le cercle-unité; un nouveau critère de familles normales ou quasi normales. C. R. Acad. Sci. Paris **236**, 1939–1941 (1953).

The author announces the following generalization of his result published in the paper reviewed above. Let

$$f(z)=c_0+c_kz^k+\cdots \quad (c_0,\ c_k\neq 0)$$

be regular in $|z|=r<1$. If $f(z)$ has defective values $\alpha_1,\cdots,\alpha_q$ different from c_0 with total defect $\geqq L>1$, then at the points of the unit disc outside circles of radius $\delta<1$ about 0, and of radii $\delta_\nu<1$ about the zeros a_ν of $f(z)$ with $|a_\nu|\leqq r$, $\log|f(z)|<(1-r)^{-3}\{H\Omega(c_0,\alpha_\lambda,\delta,\delta_\nu)+K\log 2/(1-r)\}$, where $\Omega=\sum\log(1/\delta_\nu)+\cdots$, H and K constants. Criteria of normality follow.

J. Korevaar (Madison, Wis.).

San Juan, Ricardo. L'accroissement des moments d'une fonction holomorphe dans un angle. C. R. Acad. Sci. Paris **236**, 1941–1943 (1953).

Let $f(t)$ $(t=re^{i\theta})$ be holomorphic and bounded in the sector $|\theta|\leqq\alpha\pi/2$. Suppose there is a number $C(\delta)$, independent of t and n, such that on every half-line ($\theta=$constant) of $|\theta|\leqq\alpha\pi/2-\delta<\alpha\pi/2$, the conditions

$$(1)\quad \mu_n(\theta)=\int_0^\infty|f(t)t^n|\,dt<[C(\delta)]^n\Gamma(\alpha'n+1)\qquad(n=0,1,2,\cdots)$$

hold. If $\alpha'<\alpha$, then $f(t)\equiv 0$. As it would occupy too much space to consider any of the details of the proof, we say only that it uses asymptotic series, systems of linear equations in infinitely many unknowns, and the Laplace transform.

I. M. Sheffer (State College, Pa.).

Tietz, Horst. Partialbruchzerlegung und Produktdarstellung von Funktionen auf geschlossenen Riemannschen Flächen. Arch. Math. **4**, 31–38 (1953).

On a closed Riemann surface $\mathfrak{A}$, let the analytic curve C bound the regions G and $\bar{G}$, of which $\bar{G}$ is a simply connected neighborhood of the point $\mathfrak{b}$. The author proves that any single-valued regular function $f(\mathfrak{z})$ in $G+C$ can be represented uniformly as $f(\mathfrak{z})=\sum_{n=1}^{\infty}a_n\mathfrak{E}_n(\mathfrak{z})$ where the $\mathfrak{E}_n(\mathfrak{z})$ are rational functions whose only singularities are at $\mathfrak{b}$. Here he removes the restriction placed on $\mathfrak{b}$ in his earlier paper that $\mathfrak{b}$ not be a Weierstrass point [J. Reine Angew. Math. **190**, 22–33 (1952); these Rev. **13**, 833]. The same restriction was removed by H. Roehrl [Arch. Math. **3**, 93–102 (1952); these Rev. **14**, 154]. The author then proceeds to obtain analogues of the Mittag-Leffler partial fractions expansion for meromorphic functions on $\mathfrak{A}$ and the Weierstrass infinite product expansions for functions which are entire on $\mathfrak{A}$ with a singularity at a point $\mathfrak{b}$.

G. Springer.

Umezawa, Toshio. On the multivalency of analytic functions. J. Math. Soc. Japan **4**, 279–285 (1952).

Eight theorems are established in which the conclusions are that $f(z)$ is p-valent, p-valent and star shaped, or p-valent and convex in a circle $|z|\leqq K$. The results generalise those of E. Sakai [same J. **2**, 105–113 (1950); these Rev. **12**, 601] and Z. Nehari [Amer. J. Math. **71**, 845–852 (1949); these Rev. **11**, 426] the functions concerned being meromorphic in a multiply connected domain.

A. J. Macintyre (Aberdeen).

Umezawa, Toshio. Analytic functions star-like of order p in one direction. Tôhoku Math. J. (2) **4**, 264–271 (1952).

Let $S(p)$ be the class of functions $f(z)=z^q+\sum_{q+1}^{\infty}a_nz^n$ regular for $|z|\leqq 1$, for which the image contour C of $|z|=1$ is cut by a straight line passing through the origin in exactly $2p$ points. Let $f(z)$ have q zeros at the origin and s zeros β_j, $0<|\beta_j|<1$, $j=1,2,\cdots,s$. Define t by $q+s+t=p\geqq 1$. Then $|a_n|\leqq A_n$, $n\geqq q+1$, where

$$F(z)=\frac{z^q}{(1-z)^{2q+2s}}\left(\frac{1+z}{1-z}\right)^m\prod_{j=1}^{s}\left(1+\frac{z}{|\beta_j|}\right)(1+z|\beta_j|)=z^q+\sum_{q+1}^{\infty}A_nz^n,$$

and $m=2t+1$. For the subclass of $S(p)$ for which the given straight line is parallel to the diametral line of $f(z)$, m has the value $2t$. When the coefficients a_n are all real, m may be replaced by $2[(t+1)/2]$, an earlier result due to A. W. Goodman [Proc. Amer. Math. Soc. **2**, 349–357 (1951); these Rev. **13**, 22].

M. S. Robertson (New Brunswick, N. J.).

***Conforto, Fabio. Funzioni abeliane modulari. Vol. 1. Preliminari e parte gruppale. Geometria simplettica.** Lezioni raccolte dal dott. Mario Rosati. Edizioni Universitarie "Docet", Roma, 1952. 454 pp. Lire 3900.

This is the first volume of what is to be a comprehensive account of the theory of abelian functions and their relation to the theory of generalized modular functions. The author prefaces his study of abelian functions proper with a brief discussion of elliptic functions in order to explain in this relatively simple and more familiar setting the plan of procedure and type of problem he envisages for the general case. Thus the following three aspects are emphasized. (A) The classification of elliptic fields under the identification of elliptic fields K, K' if their functions considered as dependent on the integrals of the first kind u, u' are identifiable by the relation $u'=au+b$ $(a\neq 0)$. The corresponding relations between the matrices of periods are stressed. (B) The classification of all possible sets of periods of distinct elliptic fields, that is, the construction of the fundamental domain of the restricted modular group, and the field of all modular functions. (C) The study of birational transformations on elliptic curves, the problem of division, and the function fields belonging to subgroups of the modular group. The following brief description indicates to what extent this program of generalization is carried out.

In chapter I a short account of the theory of abelian functions is presented, based upon the author's former book

"Funzioni abeliane e matrici di Riemann" [Parte 1, Univ. di Roma, 1942; these Rev. 10, 29]. It contains the classification of Riemann matrices with respect to isomorphism and the significance of the latter for the attached Picard varieties (as rational transforms). References to the work of Hurwitz, Scorza, and Albert on the existence of complex multiplications are made. In chapter II the customary concept of equivalence of Riemann matrices is modified, and it is shown how it leads to a natural (from the view point of abelian functions) introduction of symplectic groups (with elements $\begin{pmatrix} B & A \\ D & C \end{pmatrix}$). Thus, if $\omega = |\Delta^{-1}\Omega|$, Δ the matrix of elementary divisors of the given Riemann matrix, with the imaginary part of Ω being positive symmetric, the mappings $\Omega \to \Delta^{-1}(\Omega C + \Delta^{-1}D)^{-1}(\Omega A + \Delta^{-1}B)$ with the defining relations

$$B\Delta D' - D\Delta B' = C\Delta A' - A\Delta C' = 0$$

and

$$D\Delta A' - B\Delta C' = A\Delta D' - C\Delta B' = \Delta,$$

the modular group for the "level" Δ, are discussed; and it is shown that the symplectic groups do not depend, except for isomorphisms, on Δ. Next the work of Siegel on the geometry of the symplectic group is taken for a model. The symplectic cross-ratio is introduced in the space of Ω's and symplectic geometry is described in detail with the major portions of the corresponding infinitesimal geometry. Parametric representations of symplectic matrices are also studied. Siegel's theorem on the symplectic nature of the analytic transformations of the space of Ω's is presented. Chapter III is devoted to the problem of equivalence of Riemann matrices and the construction of a fundamental domain for the modular group of level Δ. The connections with the theory of quadratic forms and Eisenstein series with the prerequisite arithmetic study of matrices of integers are set forth clearly. It is to be noted that the author develops the equivalence theory of Riemann matrices for arbitrary Δ and brings out the distinction between the customary concept of equivalence. Equivalence for the level Δ requires $\omega' = \rho\omega\mathrm{T}$ for Riemann matrices of genus p, where ρ is a nonsingular complex $p \times p$ matrix and T is unimodular and satisfies $\mathrm{T}M\mathrm{T}' = M$ with $M = \begin{vmatrix} 0 & \Delta \\ -\Delta & 0 \end{vmatrix}$. It is pointed out that this distinction is especially significant for the examination of singular abelian functions.

O. F. G. Schilling.

*Szegö, G. **Conformal mapping related to torsional rigidity, principal frequency, and electrostatic capacity.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 79–83. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

Exposé de quelques inégalités "isopérimétriques", en partie nouvelles, se rattachant à des études antérieures [cf. Pólya et Szegö, Isoperimetric inequalities in mathematical physics, Princeton, 1951; ces Rev. 13, 270]. La méthode des lignes de niveau est utilisée ici sous une forme plus générale. Elle permet de montrer que pour un domaine de rayon intérieur donné en un point a, les quantités considérées sont extrémales quand le domaine est un cercle de centre a.

J. Lelong-Ferrand (Lille).

*Warschawski, S. E. **On conformal mapping of variable regions.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 175–187. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

Exposé détaillé des résultats obtenus jusqu'ici sur le problème de la représentation conforme de deux domaines plans "voisins" R_1, R_2: f_1, f_2 étant les fonctions qui représentent conformément ces domaines sur $|z| < 1$ et satisfont à $f_i(0) = 0$, $f_i'(0) > 0$ $(i = 1, 2)$ [$w = 0$ étant supposé intérieur à R_1 et R_2], on recherche des bornes pour des quantités telles que

$$|f_1 - f_2|, \quad |f_1^{(p)} - f_2^{(p)}|, \quad \int_0^{2\pi} |f_1^{(p)}(re^{i\vartheta}) - f_2^{(p)}(re^{i\vartheta})|\,d\vartheta,$$

dans a) le cas de deux domaines presque circulaires, b) le cas général. Comparaison des hypothèses et des résultats. Bibliographie étendue. *J. Lelong-Ferrand* (Lille).

*Bergman, S., and Schiffer, M. **Theory of kernel functions in conformal mapping.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 199–206. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

This address, delivered in June 1949, summarizes the main applications of the kernel function theory to conformal mapping. More inclusive expositions have meanwhile appeared, in particular, Bergman's "The kernel function . . ." [Math. Surveys, no. 5, Amer. Math. Soc., New York, 1950; these Rev. 12, 402] and Schiffer's Appendix to Courant's "Dirichlet's principle . . ." [Interscience, New York, 1950; these Rev. 12, 90]. *Z. Nehari* (St. Louis, Mo.).

*Garabedian, P. R. **A new proof of the Riemann mapping theorem.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 207–213. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

Soit D un domaine plan limité par n courbes régulières C_i de courbure bornée. L^2 désignant la classe des fonctions $f(z)$ analytiques dans D et telles que $\int\int_D |f'|^2 d\tau < \infty$, l'auteur étudie la fonction $K(z, t_1, t_2)$ qui minimise

$$A(f) = \int\int |f'|^2 d\tau / |f(t_2) - f(t_1)|^2$$

normalisée par

$$K(t_2, t_1, t_2) = \int\int_D |K'(z, t_1, t_2)|^2 d\tau, \quad K(t_1, t_1, t_2) = 0.$$

Cette fonction K est un "reproducing kernel" de S. Bergman, et permet de construire les fonctions analytiques qui représentent conformément D sur le plan coupé suivant des coupures radiales ou circulaires, les points 0, ∞ correspondant resp. à t_1, t_2.

Une deuxième méthode est fondée sur la considération de la classe $\mathfrak{L}$ des fonctions $g(z)$ définies sur $C = \sum C_i$, pas nécessairement uniformes, mais à variation totale bornée et satisfaisant à $\int_C (z - w)^{-1} dg = 0$ pour tout w extérieur à D. Soit $l(z, t)$ la fonction de classe $\mathfrak{L}$ qui minimise $\int_C |dg|$, normalisée par $\int_C (z - t)^{-1}\, dg = 2i\pi$ $(t \in D)$. Cette fonction permet de construire diverses représentations conformes de D, non toutes explicitées. Cette méthode se rattache à une

étude antérieure de l'auteur [Duke Math. J. 16, 439–459 (1949); ces Rev. 11, 21].

Signalons qu'il existait déjà une application des noyaux de Bergman due à Z. Nehari [ibid. 16, 165–178 (1949); ces Rev. 10, 440] à la construction de représentations canoniques; mais les deux méthodes sont essentiellement différentes. *J. Lelong-Ferrand* (Lille).

***Nehari, Zeev. The kernel function and the construction of conformal maps.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 215–224. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

L'auteur a déterminé antérieurement [Duke Math. J. 16, 165–178 (1949); ces Rev. 10, 440], au moyen des noyaux de Bergman, les fonctions qui représentent conformément un domaine D, limité par n courbes analytiques, sur les domaines canoniques de Koebe. Il montre ici que chacune de ces fonctions est associée à un problème d'extremum, et il obtient ainsi des inégalités en sens inverse de celles qui résultent des propriétés de minimum de l'intégrale de Dirichlet, ce qui fournit une estimation des constantes associées aux représentations considérées.

J. Lelong-Ferrand (Lille).

***Ahlfors, Lars V., and Beurling, A. Conformal invariants.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 243–245. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

Exposé d'une méthode d'investigation qui s'apparente à l'ancien lemme de Grötzsch, mais d'une portée beaucoup plus générale. L'invariant conforme considéré ici est la longueur extremale $\lambda\{\gamma\}$ d'une famille de courbes $\{\gamma\}$ contenues dans un domaine Ω. Par définition:

$$\lambda\{\gamma\} = \sup [L_\rho\{\gamma\}]^2/A_\rho(\Omega)$$

pour toutes fonctions $\rho \geqq 0$ dans Ω avec

$$L_\rho = \inf_{\gamma \in \{\gamma\}} \int_\gamma \rho |dz|, \quad A_\rho = \int\int_\Omega \rho^2 dxdy.$$

Une quantité analogue avait déjà été définie par J. Leray dans un cas particulier [Comment. Math. Helv. 8, 149–180 (1936), p. 173] et appliquée à la théorie des sillages; mais les travaux ultérieurs de J. Leray n'ont pas été publiés.

La définition actuelle a été présentée par les auteurs au Congrès Scandinave de 1946 et appliquée par eux dans un cas particulier [cf. Ahlfors et Beurling, Acta Math. 83, 101–129 (1950), p. 114; ces Rev. 12, 171]. Une définition voisine a été utilisée depuis par J. Hersch [C. R. Acad. Sci. Paris 235, 569–571 (1952); ces Rev. 14, 262].

J. Lelong-Ferrand (Lille).

Schild, Albert. On a problem in conformal mapping of schlicht functions. Proc. Amer. Math. Soc. 4, 43–51 (1953).

Soit $w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ représentant conformément $|z| < 1$ sur un domaine D étoilé par rapport à $w = 0$; soit r_0 la borne supérieure des r tels que l'image de $|z| < r$ soit un domaine convexe. On désigne par d^* la distance de $w = 0$ à la frontière de D, par d_0 la distance de $w = 0$ à l'image de $|z| = r_0$. La limite inférieure exacte de r_0 est $2\text{–}3^{1/2}$. On a émis l'hypothèse que celle de d_0/d^* est 2/3. L'auteur étudie la borne inférieure de d_0/d^* pour les classes de fonctions suivantes: (a) $f(z) = z + \sum_{n=1}^{\infty} a_{np+1} z^{np+1}$ $(p = 2, 3, \cdots)$; (b) a_n réel $(n \geqq 2)$ et diverses hypothèses géométriques sur la frontière de D. L'hypothèse $d_0/d^* \geqq 2/3$ est démontrée dans quelques cas. *J. Lelong-Ferrand* (Lille).

af Hällström, Gunnar. Eine Bemerkung über Einschnittgebiete. Acta Acad. Aboensis 18, no. 6, 7 pp. (1952).

The author continues his study [Soc. Sci. Fenn. Comment. Phys.-Math. 16, no. 13 (1952); these Rev. 14, 549] of the boundary correspondence in conformal mappings of the unit circle onto a circle with a countable infinity of radial slits l_n ending on the circumference. The present paper devotes particular attention to the case in which the set of limit points of the images λ_n of l_n is of positive capacity.

Z. Nehari (St. Louis, Mo.).

Unkelbach, Helmut. Über die Approximation schlichter konformer Abbildungen durch kongruente Iterationen spezieller sterniger oder konvexer konformer Abbildungen. Math. Z. 58, 63–70 (1953).

Let B denote the set of functions $f(z), f(0) = 0$, which are regular, univalent, and satisfy $|f(z)| < 1$ in $|z| < 1$. A function of the form $f_n(f_{n-1}(\cdots(f_1(z))\cdots))$ is called a congruent iterate of $f(z)$ if for $\nu = 1, 2, \cdots, n$, $f_\nu(z) = e^{i\psi_\nu} f(e^{i\theta_\nu} z)$, where ψ_ν, θ_ν are real. The author proves that each $F(z) \in B$ can be approximated by a suitable sequence of congruent iterates of functions $S_k(z)$ taken from a certain sequence whose precise definition is too long to reproduce here. This result is applied to the problem of bounding $|F'(z)|$ for $F(z) \in B$, and in the subset of convex functions from B.

A. W. Goodman (Lexington, Ky.).

Myrberg, P. J. Über die Linearisierung der schlichten konformen Abbildungen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 145, 8 pp. (1953).

Let D be a plane domain and $z' = T(z)$ a one-to-one conformal mapping of D into itself. The author proves that if there are two boundary points which remain fixed under T, then there is a conformal mapping $w = \varphi(z)$ of the domain D onto a domain D' such that on D' the transformation $\varphi T \varphi^{-1}$ is a linear transformation. An example is given to show that φ need not be unique to within a linear transformation.

H. L. Royden (Stanford, Calif.).

***Bolder, Harm. Flow functions and their application to some problems of conformal mapping.** Thesis, University of Leiden, 1953. 38 pp.

Proof of the existence of the conformal mapping of a simply-connected domain onto a rectangle, based directly on the Dirichlet principle, followed by a refinement of the Ahlfors distortion theorem. *P. R. Garabedian.*

Paatero, V. Über die konforme Abbildung mehrblättriger Gebiete von beschränkter Randdrehung. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 128, 14 pp. (1952).

Let $\zeta(z)$ map $|z| < 1$ onto a domain of bounded boundary rotation on a finitely-sheeted Riemann surface. The author first proves the representability of $1 + z\zeta''(z)/\zeta'(z)$ in the form of a Poisson-Stieltjes integral. He then shows conversely that any function given by the integral formula may be regarded as a functional of this type. A generalization is stated to the case where the image-domain contains infinitely many branch-points. *Y. Komatu* (Tokyo).

Volkovyskiĭ, L. I. Quasi-conformal mappings and problems on conformal pasting. Ukrain. Mat. Žurnal **3**, 39–51 (1951). (Russian)

Soit $\phi(x)$ une fonction continue définie sur $[-1, 1]$, strictement croissante, avec $\phi(-1)=-1$, $\phi(1)=1$. On dit que les deux demi-disques $D_1(|z|<1,\ y>0,\ z=x+iy)$, $D_2(|z|<1,\ y<0)$ admettent un collage conforme correspondant à la fonction ϕ, s'il existe deux fonctions $\omega=f_1(z)$, $\omega=f_2(z)$ représentant conformément D_1 et D_2 respectivement sur D_1' et D_2', ces deux derniers domaines étant deux parties du disque $|\omega|<1$, séparées par un arc de Jordan simple γ, le diamètre $(-1, 1)$ des domaines D_1, D_2 étant transformé en γ, avec $f_1(x)=f_2[\phi(x)]$ pour tout $x \in (-1, 1)$. Ce collage est dit complètement déterminé s'il est déterminé à une transformation linéaire de $|\omega|<1$ en lui-même, près. Si $\phi(x)$ admet une dérivée positive continue telle qu'il existe une constante K pour laquelle

$$\int_{-1}^{1} |(\phi'(x)-\phi'(t))/(x-t)|\,dx<K \quad (t \in [-1, 1]),$$

alors il existe un collage parfaitement déterminé correspondant à ϕ, les fonctions correspondantes $f_1(z)$ et $f_2(z)$ admettant des dérivées $f_1'(x)$, $f_2'(x)$ différentes de zéro, bornées sur $[-1, 1]$. D'autres théorèmes avec des conditions moins restrictives sur ϕ sont donnés. On indique ensuite des propriétés différentielles d'une représentative quasi-conforme. *S. Mandelbrojt* (Houston, Tex.).

Heins, Maurice. Studies in the conformal mapping of Riemann surfaces. I. Proc. Nat. Acad. Sci. U. S. A. **39**, 322–324 (1953).

Let F and G be hyperbolic Riemann surfaces with the Green's functions g_F and g_G, respectively. Consider a conformal mapping $f(p)$ of F into G, and let $n(p)$ denote the multiplicity of f at $p \in F$. The Lindelöf inequality asserts that, for each $q \in G$, the term u_q in

$$g_G[f(p), q]=\sum_{f(r)=q} n(r)g_F(p, r)+u_q(p)$$

is non-negative harmonic on F. Consequently, u_q has a unique decomposition into a quasi-bounded harmonic function v_q and a singular harmonic function [Parreau, Ann. Inst. Fourier Grenoble **3** (1951), 103–197 (1952); these Rev. **14**, 263]. The author announces the following results. (1) $(p, q)\to u_q(p)$ is upper semi-continuous, $(p, q)\to v_q(p)$ lower semi-continuous on $F\times G$. (2) Either $v_q=0$ for all q or else for no q. (3) The set of q for which $u_q-v_q>0$ is of capacity zero.

The author terms f of type-Bl (Blaschke) if $v_q=0$ for all q. A number of theorems concerning f are given, a typical one being as follows: If f is, in a specified sense, locally of type-Bl, then it is of type-Bl. Several applications are indicated. The paper contains no proofs. *L. Sario.*

Ohtsuka, Makoto. Note on the harmonic measure of the accessible boundary of a covering Riemann surface. Nagoya Math. J. **5**, 35–38 (1953).

Cette note apporte des précisions à des inégalités déjà démontrées par l'auteur [Nagoya Math. J. **4**, 109–118 (1952); ces Rev. **14**, 36, désigné ci-dessous par (M)]. Soit R une surface de Riemann de frontière positive étalée sur une surface de Riemann $\mathbf{R}$. Soient C les chemins sur R qui tendent vers un point de la frontière idéale de R, leur projection sur $\mathbf{R}$ tendant vers un point de $\mathbf{R}$; $v(P)$ les fonctions harmoniques continues sur R, telles que $0\leqq v\leqq 1$, $\lim v(P)=1$ le long des chemins C; $\mu(P, R)$ l'enveloppe inférieure des $v(P)$, qui est harmonique d'après le principe de Perron-Brelot. Désignons par l'indice ∞ le recouvrement universel d'une surface. 1) Soit $\mathbf{R}'$ la projection de R sur $\mathbf{R}$. Supposons que $(\mathbf{R}')^\infty$ ne soit pas de type hyperbolique, mais que R ne couvre des points P_n de $\mathbf{R}$ en nombre fini qu'un nombre fini de fois, de sorte que le recouvrement universel $(\mathbf{R}-\{P_n\})^\infty$ sont de type hyperbolique. Soit $\bar{R}$ la surface obtenue en retirant de R les points qui ont pour supports les P_n. Il résultait de (M): $\mu(P, R)\geqq\mu(P, \bar{R}^\infty)$. L'auteur démontre ici l'égalité. 2) Supposons que R soit un domaine de $\mathbf{R}$ avec frontière positive. Il résultait de (M): $\mu(P, R)\geqq\mu(P, R^\infty)$. L'auteur démontre ici l'égalité. La démonstration fait intervenir le principe de Perron-Brelot. *R. de Possel.*

Huber, Heinz. Über analytische Abbildungen Riemannscher Flächen in sich. Comment. Math. Helv. **27**, 1–73 (1953).

This thesis is a comprehensive study of analytic mappings A of a Riemann surface R into itself. The starting point is the following earlier result of the author [Compositio Math. **9**, 161–168 (1951); these Rev. **13**, 337]: Every A of an annulus $R: 0<r_1<|z|<r_2$ which maps all not nullhomotopic closed curves onto such curves is an automorphism (conformal self-mapping). The main theorem states that this result remains valid for a class R_D of Riemann surfaces characterized by the author as having a "discrete modular spectrum".

The following concepts are introduced. Let G be a Riemann surface of hyperbolic type (the author does not refer to surfaces which possess Green's functions, but to surfaces whose universal covering surface has this property). A hyperbolic Poincaré metric is imposed upon G, and the hyperbolic length of a closed curve $p(t)$, $0\leqq t\leqq 1$, on G is denoted by $\mu[p(t)]$. For every homotopy class W of closed curves, a nonnegative number, the modulus $M[W]$ of W, is defined to be

$$M[W]=\inf_{p(t)\in W} \mu[p(t)].$$

If $M[W]>0$, then the author calls W hyperbolic, otherwise parabolic. G is said to have a discrete modular spectrum if, for every $m>0$, the number of classes W with $0<M[W]<m$ is finite. The boundary of a subregion G of a Riemann surface R is called normal if each boundary point a of G is either isolated or has a neighborhood $N_a\subset R$ such that each closed curve $p(t)$ in $N_a\cap G$ is nullhomotopic.

Using these notions, the following theorems are established, G being a "hyperbolic" region with compact closure on R: (1) A class W on G is parabolic if and only if a curve $\gamma \in W$ can be reduced to an isolated boundary point $a \in R$ of G. (2) If the boundary of G is normal, then $G \in R_D$. (3) Every "hyperbolic" Riemann surface with a finite Betti number belongs to R_D. The proofs of these theorems are based on generalizations of Picard's theorem on omitted values and Montel's theorem on continuously converging sequences.

Without the assumption of a discrete modular spectrum, the following statements are proved for Riemann surfaces R whose fundamental group is nonabelian: (1) The group of all analytic automorphisms of R which leave unchanged a class W different from the nullclass is a cyclic group of finite order. (2) If an A of R carries a hyperbolic curve $p(t)$ onto a homotopic curve $A(p(t))$, then A is a periodic automorphism.

For surfaces $R \in R_D$, the main results are as follows: (1) If the fundamental group of R is nonabelian, then the group

of all analytic automorphisms of R is of finite order. (2) If R contains no parabolic classes W, then there exists, for each $p(t)$, an integer $n>1$, which depends solely on the class W of $p(t)$, and has the property that for every non-automorphic A of R the curve $A^n(p(t))$ is homotopic to zero. (3) Any A of R which carries each closed not nullhomotopic curve to another such curve is an automorphism. Further theorems are derived for "analytic nullhomotopic mappings".

L. Sario (Cambridge, Mass.).

*Sario, Leo. **Sur la classification des surfaces de Riemann.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 229–238. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

This paper is concerned with the classification problem for non-compact Riemann surfaces. It will be convenient to introduce the following notation in order to state the principal results. Let F denote a non-compact Riemann surface, H the class of non-constant, single-valued harmonic functions on F, B the class of bounded functions on F, D the class of functions on F with bounded Dirichlet integral, M the class of functions on F with bounded Carleman-Nevanlinna mean [cf. Nevanlinna, Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **57** (1949); these Rev. **11**, 516]. Let HK ($K=B, D, M$) denote the intersection of H and K. Let O_{HK} denote the class of surfaces for which HK is empty. The classes O_{HK} are termed absolute. Corresponding relative classes are introduced and consist of surfaces not admitting in a neighborhood of the ideal boundary harmonic functions which are non-constant, vanish on the relative boundary of the neighborhood and satisfy condition K (with respect to the neighborhood of the ideal boundary in question). Similarly, corresponding classes of analytic functions AK are introduced and also the relative classes A_0K where A_0 consists of the class of analytic functions non-constant, and single-valued in a neighborhood of the ideal boundary with real part vanishing on the relative boundary of the neighborhood. O_{AK} is then the class of surfaces not admitting any function in AK and similarly O_{A_0K} is the class of surfaces for which there does not exist any function of class A_0K.

The principal results are the following: (1) If $K=B, M, D$, then O_{H_0K} is the class of parabolic surfaces (i.e. without Green's functions) and $O_{H_0K}\subseteq O_{HK}$. For surfaces of finite genus $O_{HK}=O_{H_0K}$. (2) For the above K, $O_{AK}\supseteq O_{HK}$ (strict inclusion for $K=B, D$). For surfaces of finite genus $O_{A_0K}=O_{AK}$ for all K.

The proofs depend upon the use of harmonic measure, relations between the Dirichlet integral and the Carleman-Nevanlinna mean, and the Schwarz alternating method to construct harmonic functions with assigned singular behavior.

M. Heins (Providence, R. I.).

Sario, Leo. An extremal method on arbitrary Riemann surfaces. Trans. Amer. Math. Soc. **73**, 459–470 (1952).

Ce mémoire contient de nombreux résultats démontrés par des moyens élémentaires et rapides. Les propriétés extrémales obtenues comprennent comme cas particuliers des résultats de: de Possel [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1931**, 199–202; Math. Ann. **107**, 496–504 (1932)], Grötzsch [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. **84**, 15–36 (1932)]; Grunsky [Schr. Math. Sem. Inst. Angew. Math. Univ. Berlin **1**, 95–140 (1932)]; Schiffer [Duke Math. J. **10**, 209–216 (1943); ces Rev. **4**, 271]. Les applications de ces propriétés à la classification des surfaces de Riemann étendent largement des résultats de Schiffer [loc. cit.], Lehto [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **67** (1949); ces Rev. **11**, 338], Ahlfors et Beurling [Acta Math. **83**, 101–129 (1950); ces Rev. **12**, 171], Parreau [C. R. Acad. Sci. Paris **231**, 751–753 (1950); ces Rev. **12**, 402], Virtanen [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **75** (1950); ces Rev. **12**, 403], Royden [ibid. no. **85** (1951); ces Rev. **13**, 339], Lokki [ibid. no. **92** (1951); ces Rev. **13**, 338]. L'auteur définit sur une surface de Riemann quelconque une classe de fonctions (caractérisée par $\Delta(p)\leqq 0$, voir ci-dessous) qui joue un rôle analogue à celui des fonctions univalentes pour les surfaces planaires. Enumérons les principaux résultats.

I. Soit R une surface de Riemann ouverte, β sa frontière idéale, $D(u)$ l'intégrale de Dirichlet sur R de la fonction réelle u, $|z|<1$ un disque fixe image conforme d'un domaine de R, $P=p+ip^*$ une fonction analytique sur R, de partie réelle p uniforme, sans singularité sauf en $z=0$ où l'on a $P=1/z^k+\sum_1^\infty a_\nu z^\nu$, a_ν la partie réelle de a_ν, λ un nombre réel tel que $-1\leqq\lambda\leqq 1$, $\Delta(P)=\int_\beta p\,dp^*$.

Parmi les P, il en existe une et une seule $P_\lambda=p_\lambda+ip_\lambda^*$ qui minimise $m_\lambda(p)=\lambda\cdot 2\pi k a_k+\Delta(P)$. Elle a les propriétés suivantes:

(1) $$m_\lambda(p)=m_\lambda(p_\lambda)+D(p-p_\lambda);$$

(2) $$m_\lambda(p_\lambda)=2^{-1}\pi k[(1+\lambda)^2\alpha_{1,k}-(1-\lambda)^2\alpha_{-1,k}];$$

(3) $$2P_\lambda=(1+\lambda)P_1+(1-\lambda)P_{-1};$$

(4) $\Delta(P_\lambda)=2^{-1}\pi k(\lambda^2-1)\sigma\leqq 0$ où $\sigma=\alpha_{-1,k}-\alpha_{1k}$; (5) $\Delta(P)\leqq 0$ entraîne $\alpha_{1k}\leqq\alpha_k\leqq\alpha_{-1,k}$; (6) l'identité de toutes les p telles que $\Delta(P)\leqq 0$ équivaut à $\sigma=0$; (7) $u=p_{-1}-p_1$ est bornée sur R et minimise $D(u)$ pour toutes les fonctions réelles normalisées, la minimum étant $2\pi k\sigma^{-1}$.

II. Soit $Q=q+iq^*$ une P telle que la période de $\bar p$ s'annule le long de tout cycle qui partage R. Il existe une Q et une seule, soit Q_λ, qui minimise $m_\lambda(q)$. Les résultats de I restent valables en remplaçant P_λ par Q_λ; σ est remplacé par τ.

Supposons R planaire. Quand Q est univalente et représente R sur un domaine plan, désignons par A l'aire du complémentaire. Alors: (1) Q_1 représente R sur un domaine à fentes verticales et réalise $\min(2\pi\alpha_1-A)=2\pi\alpha_{11}$; de même pour Q_{-1} avec fentes verticales et réalisation de

$$\max(2\pi\alpha_1+A)=2\pi\alpha_{-1,1}.$$

(2) Q_0 réalise $\min\Delta(Q)=\Delta(Q_0)=-2^{-1}\pi\tau$. (3) Parmi les G analytiques et uniformes sur R telles que $G=\sum_1^\infty b_\nu z^\nu$, $\mathrm{Re}(b_1)=1$, la fonction $G_0=\tau^{-1}(Q_{-1}-Q_1)$ réalise

$$\min D(G)=D(G_0)=2\pi k\tau^{-1}.$$

Comme conséquences, les conditions suivantes sont équivalentes pour les surfaces planaires: (a) $\tau=0$, (b) R est "rigide", (c) on n'a jamais $\Delta(Q)<0$, (d) $A=0$ pour toute représentation univalente de R.

III. L'auteur introduit les notations suivantes: O_P est la classe des surfaces sur lesquelles il n'existe pas de fonction de class F, S_σ et S_τ sont les classes de surfaces pour lesquelles $\sigma=0$ et $\tau=0$. FG est la class intersection des classes F et G. H est la classe des fonction harmoniques uniformes non constantes, K la sous-class de H pour laquelle la partie imaginaire a une période nulle le long de tout cycle qui partage, A celle des fonctions analytiques non constantes, W des méromorphes non constantes, S des univalentes uniformes non constantes, B des bornées, D des fonctions dont l'intégrale de Dirichlet est finie, E de celles qui

omettent un ensemble d'aire positive. On a alors

(1) $O_{HB} \subset O_{HBD} = O_{HD} = S_r$,

(2) $O_{KB} \subset O_{KBD} = O_{KD} = S_r$,

(3) $O_{WB} = O_{AB} \subset O_{AD}$.

Pour des surfaces planaires: $O_{AB} \subset O_{ABD} = O_{AD} = O_{BE} = S_r$.

R. de Possel (Alger).

Tôki, Yukinari. On the classification of open Riemann surfaces. Osaka Math. J. 4, 191–201 (1952).

The author succeeds in solving the long disputed question as to whether or not the inclusions $O_G \subset O_{HB} \subset O_{HD}$ are strict; the answer is in the affirmative. Here the symbols O_F stand for Riemann surfaces with F-removable boundary, that is, surfaces which share with closed surfaces the property of not admitting single-valued functions of a given class F. The classes G, HB, HD consist of Green's functions, nontrivial harmonic bounded functions, and harmonic functions with a finite Dirichlet integral, respectively.

The evolution of the problem is illustrative of its difficulty. To motivate his claim that the classical theory of Abelian integrals could be extended to parabolic surfaces only, Nevanlinna made the assertion that $O_G = O_{HD}$ [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 1 (1941), see p. 31, first paragraph; these Rev. 7, 427]. This equality, together with $O_G = O_{HB}$, was claimed also by Virtanen [ibid. no. 56 (1949); these Rev. 11, 510]. In the special case of finite genus or, for arbitrary surfaces, of relative removability, both statements are correct [see the paper reviewed second above]. In the general case, however, the inclusions $O_G \subseteq O_{HB} \subseteq O_{HD} (= O_{HBD})$ were proved by various methods [Virtanen, Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 75 (1950); these Rev. 12, 403; Royden, ibid. no. 85 (1951); these Rev. 13, 339; Sario, the paper reviewed above]. Ahlfors was the first to construct an example to show that $O_G \subset O_{HD}$ is strict [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 87 (1951); these Rev. 13, 338]. Although he later announced a technical inaccuracy [these Rev. 13, 338], his basic idea led to a rigorous proof [Ahlfors and Royden, see the paper reviewed below].

The present author constructs two examples, one for proving $O_G \subset O_{HB}$, the other $O_{HB} \subset O_{HD}$, strictly. His reasoning, with involved diagrams, figures, tables and schemes, is somewhat discouraging to the reader. Also, at some points, the presentation is not accurate. E.g., on p. 195, the function $T_m(z) = [T(z^{2^m})]^{1/2^m}$ could reduce to $[\bar{z}^{2^m}]^{1/2^m} = \bar{z}$, and the argument would collapse. On p. 197, the first inequality does not hold for the harmonic measure of a slit, as stated by the author. Such errors in presentation do not, however, invalidate his construction. A simpler presentation could be given in full as follows.

To prove that $O_G \subset O_{HB}$, consider the disc $|z| < 1$ with the radial slits

$$1 - \frac{1}{2\mu} \leqq |z| \leqq 1 - \frac{1}{2\mu+1}, \quad \arg z = \nu \cdot \frac{2\pi}{2^{\mu+1}},$$

where $\mu = 1, 2, \cdots$ and $\nu = 1, \cdots, 2^{\mu+1}$. Write $\mu = 2^{m-1}(2n-1)$ with $m, n = 1, 2, \cdots$, and denote the slits by C^{ν}_{mn}. For each (fixed) m, identify, by pairs, the edges of those C^{ν}_{mn} which lie in the same sector S_{mh}:

$$(h-1)\frac{2\pi}{2^{m+1}} \leqq \arg z \leqq h\frac{2\pi}{2^{m+1}},$$

with $h = 1, \cdots, 2^{m+1}$, and are symmetrically placed with respect to the bisecting half ray d_{mh} of S_{mh}. On the Riemann surface W thus constructed, let u be a function of class HB, $|u| \leqq 1$. For a fixed m and for $z \in W$, take the point z_m lying in the same S_{mh}, symmetrically placed with respect to d_{mh}, and consider the function $u_m(z) = \frac{1}{2}[u(z) - u(z_m)]$. As it vanishes on those C^{ν}_{mn} which lie in the annulus R_{mn}:

$$1 - \frac{1}{2\mu-1} \leqq |z| \leqq 1 - \frac{1}{2\mu+2},$$

it is dominated by the harmonic measure ω_{mn} of the boundary of R_{mn} with respect to the region $R_{mn} - \bigcup_{\nu} C^{\nu}_{mn}$. Since $\omega_{mn} \to 0$ for $n \to \infty$ uniformly on the circles through the centers of C^{ν}_{mn}, it follows that $u(z) \equiv u(z_m)$. This yields, for $m \to \infty$, the required HB-removability relation $u(z) = \text{const.}$ On the other hand, the Green's function $u = \log(1/|z|)$ exists on W, and the strictness of $O_G \subset O_{HB}$ follows.

For the proof of the strict inclusion $O_{HB} \subset O_{HD}$, consider the annulus $\frac{1}{2} < |z| < 2$ with the slits D^{ν}_{mn}:

$$\tfrac{1}{2} + \frac{1}{2\mu+1} \leqq |z| \leqq \tfrac{1}{2} + \frac{1}{2\mu},$$

$$2 - \frac{1}{2\mu} \leqq |z| \leqq 2 - \frac{1}{2\mu+1},$$

$$\arg z = \nu \cdot \frac{2\pi}{2^{\mu+1}}.$$

Take an infinite collection $\{F(k)\}$, $k = 1, 2, \cdots$, of copies of these slit annuli F, and, successively for each (fixed) $m = 1, 2, \cdots$ and $j = 0, 1, \cdots$, join $F(i + 2^m j)$ with $F(i + 2^{m-1} + 2^m j)$, $i = 1, \cdots, 2^{m-1}$, along the edges of $\bigcup_{\nu,n} D^{\nu}_{mn}$, with a folding at each edge. For each p on the resulting Riemann surface W, take the point p_m corresponding to p if the copies of F joined along D^{ν}_{mn} are interchanged. If $u(p) \in HBD$ on W, the function $u_m(p) = \frac{1}{2}[u(p) - u(p_m)]$ vanishes on these slits, and one concludes again that $u(p) \equiv u(p_m)$. For $m \to \infty$ it follows that $u(p)$ takes on identical values on all copies of F. This gives, the Dirichlet integral being finite, the HD-removability relation $u(p) = \text{const.}$ The proof is completed by the fact that $\log|z| \in HB$ on W.

L. Sario.

Ahlfors, Lars V., and Royden, H. L. A counterexample in the classification of open Riemann surfaces. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 120, 5 pp. (1952).

Let O_G and O_{HD} be the classes of Riemann surfaces with G- and HD-removable boundaries, respectively. To prove that the inclusion $O_G \subset O_{HD}$ is strict, the authors construct a hyperbolic Riemann surface without HD-functions as follows. On $|z| < 1$, take the slits Δ_n^h: $r_n \leqq |z| \leqq r_{n+1}$, $\arg z = h \cdot 2\pi/2^n$ with $h = 0, 1, \cdots, 2^n - 1$. Divide each Δ_n^h into 2^{n-1} subslits Δ_n^{hk} of equal logarithmic length, and identify the left edge of Δ_n^{hk} with the right edge of $\Delta_n^{h+k,k}$. Then $\log(1/|z|)$ is the Green's function on the resulting Riemann surface, while every HD-function is proved to have, for $\{r_n\}$ suitably chosen, constant radial limits and, consequently, reduce to a constant.

The sharper result that even the inclusions $O_G \subset O_{HB} \subset O_{HD}$ are strict was independently (with slight priority) obtained by Tôki [see the paper reviewed above].

L. Sario.

Royden, H. L. Some counterexamples in the classification of open Riemann surfaces. Proc. Amer. Math. Soc. 4, 363–370 (1953).

Let O_{HB} and O_{HD} be the classes of Riemann surfaces with HB- and HD-removable boundaries, respectively. The

author shows that the inclusion $O_{HB} \subset O_{HD}$ is strict [cf. the second preceding review]. He establishes, moreover, that there are Riemann surfaces on which the spaces *HD* and *HBD* have dimension n. A bounded Riemann surface is also constructed which has a nonconstant *HD* function whose normal derivative vanishes on the boundary while every *HD* function with vanishing boundary values is constant. *L. Sario* (Cambridge, Mass.).

Igusa, Jun-ichi. On a property of the domain of regularity. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 95–97 (1952).

Es bezeichne $\mathfrak{D}$ ein schlichtes Regularitätsgebiet im offenen Raum C^n von n komplexen Veränderlichen; es sei $\mathfrak{O}$ der Integritätsring der in $\mathfrak{D}$ regulären Funktionen. Es werden die maximalen Ideale $\mathfrak{P}$ von $\mathfrak{O}$ untersucht. Der Restklassenring $\mathfrak{O}/\mathfrak{P}$ enthält offenbar den komplexen Körper C^1. Ist P ein beliebiger Punkt von $\mathfrak{D}$, so bildet die Menge aller in $\mathfrak{D}$ regulären, in P verschwindenden Funktionen ein maximales Ideal $\mathfrak{P}$ von $\mathfrak{O}$, und es gilt $\mathfrak{O}/\mathfrak{P} = C$. Verf. beweist mittels der Cartan-Okaschen Idealtheorie die Umkehrung: Ist $\mathfrak{P}$ ein maximales Ideal von $\mathfrak{O}$ und gilt $\mathfrak{O}/\mathfrak{P} = C$, so gibt es einen Punkt $P(a_1, \cdots, a_n) \in \mathfrak{D}$, so dass für jedes $f \in \mathfrak{P}$ gilt: $f(P) = 0$. Die in P lokalen Koordinaten $z_\nu - a_\nu$ $(1 \leqq \nu \leqq n)$ bilden eine Basis von $\mathfrak{P}$. Umgekehrt bildet jede Basis von $\mathfrak{P}$ der Länge n ein lokales Koordinatensystem in P. Zu verschiedenen maximalen Idealen $\mathfrak{P}_1$, $\mathfrak{P}_2$ mit $\mathfrak{O}/\mathfrak{P}_1 = \mathfrak{O}/\mathfrak{P}_2 = C$ gehören verschiedene Punkte P_1, P_2 von $\mathfrak{D}$. Daraus folgt: Die Menge $\hat{D}$ aller im Integritätsring $\mathfrak{O}$ maximalen Ideale $\mathfrak{P}$ mit $\mathfrak{O}/\mathfrak{P} = C$ wird zu einer n-dimensionalen komplexen Mannigfaltigkeit, wenn man jedem $\mathfrak{P} \in \hat{D}$ als lokale Koordinaten eine Basis von $\mathfrak{P}$ der Länge n zuordnet. Die so gewonnene komplexe Mannigfaltigkeit $\hat{D}$ ist analytisch isomorph zu $\mathfrak{D}$. *H. Behnke* (Münster).

Bergman, S., and Schiffer, M. Potential-theoretic methods in the theory of functions of two complex variables. Compositio Math. **10**, 213–240 (1952).

The authors investigate the space of functions of two complex variables by associating with each element of this space certain quantities, arising from potential-theoretic considerations, which describe certain properties of the manifolds met with in the geometry of these functions. The main part of the paper is devoted to the derivation of inequalities connecting the various quantities concerned. The authors also generalize, to the case of functions of two complex variables, the relations between the kernel function and the Green's functions established in their earlier work on partial differential equations [Duke Math. J. **14**, 609–638 (1947); these Rev. **9**, 187]. *Z. Nehari* (St. Louis, Mo.).

Wille, R. J. Sur la transformation intérieure d'une surface non orientable sur le plan projectif. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 63–65 (1953).

The author proves the following theorem. Given a 2-manifold M with a countable basis, orientable or not, there exists an interior mapping of M into the projective plane. This theorem extends a result of Stoïlow [Leçons sur les principes topologiques de la théorie des fonctions analytiques, Gauthier-Villars, Paris, 1937, pp. 77, 119], also proved by M. Heins [Proc. Amer. Math. Soc. **2**, 951–952 (1951); these Rev. **13**, 547], that there is an interior mapping of an orientable M into the sphere. *C. J. Titus.*

Fréchet, Maurice. Sur les fonctions paraanalytiques à deux et à trois dimensions. C. R. Acad. Sci. Paris **236**, 348–351 (1953).

In a preceding note [same C. R. **235**, 1585–1587 (1952); these Rev. **14**, 463] the author defined two so-called canonical forms D and P of families of functions "paraanalytiques". In this note he obtains, for example, the result that the families D and P form a tri-operational algebra (addition, multiplication, composition) in the sense of Menger. He gives an example of an extension of functions "paraanalytiques" to three dimensions. *C. J. Titus.*

Theory of Series

Zamansky, Marc. Sur la sommation des séries divergentes. C. R. Acad. Sci. Paris **235**, 1094–1096 (1952).

In a previous paper [same C. R. **233**, 999–1001 (1951); these Rev. **13**, 455] the author has defined a special method of summation of series. Let $g(x)$ be a polynomial with $g(0) = 1$ and having a zero of order p at $x = 1$. Given a series $\sum u_k$, set $T_x(g) = \sum_{k \leqq x} g(k/x) u_k$. Then, $\sum u_k$ is summable (g) to L if $\lim_{x \to \infty} T_x(g) = L$. In this, x may also be allowed to take only integral values. Calling this more general method $(g)'$ the author states that summability (g) is equivalent to summability $(g)'$ plus the condition $u_n = o(n^p)$. In a previous paper [ibid. **234**, 1025–1027 (1952); these Rev. **13**, 737] the author and H. Delange gave conditions under which (g) is equivalent to (C, p). It is now asserted that if $p = 0$ and $\sum u_n$ is summable (g), then $\sum u_n$ is either convergent, or $\sum_1^n u_k$ differs from a convergent sequence by a linear combination of the terms $n^\alpha (\log n)^\beta$, or else $\sum u_n$ is summable by an iterated logarithmic mean. *R. C. Buck.*

Meyer-König, Werner. Bemerkung zu einem Lückenumkehrsatz von H. R. Pitt. Math. Z. **57**, 351–352 (1953).

Let λ_n be a sequence of nonnegative integers such that, for some positive constant λ, $\lambda_{n+1} - \lambda_n \geqq \lambda(\lambda_n)^{1/2}$. Let $\sum a_n$ be a series of complex terms for which $a_n = 0$ when $n \neq \lambda_0, \lambda_1, \lambda_2, \cdots$. Let $\sum a_n$ be evaluable by the Borel exponential method B. A Tauberian theorem of Pitt [Proc. London Math. Soc. (2) **44**, 243–288 (1938); Theorem 17] then implies that if, for each $\alpha > 1$, $a_n = O(\alpha^n)$, then $\sum a_n$ converges. This fact and further arguments are used to prove that $\sum a_n$ converges if there exists at least one $\alpha > 1$ such that $a_n = O(\alpha^n)$. *R. P. Agnew* (Ithaca, N. Y.).

Peyerimhoff, Alexander. Untersuchungen über absolute Summierbarkeit. Math. Z. **57**, 265–290 (1953).

This paper discusses absolute summability factors ϵ_ν using methods which proved successful for ordinary summability factors [Peyerimhoff, Math. Z. **55**, 23–54 (1951); Jurkat and Peyerimhoff, ibid. **55**, 92–108 (1951); these Rev. **13**, 933, 934]. There are summability factors of two kinds: (i) the series $\sum s_\nu \epsilon_\nu$ is B summable for each $|A|$ summable sequence s_ν; (ii) $\sum a_\nu \lambda_\nu$ is B summable for each $|A|$ summable series $\sum a_\nu$. Almost all conditions for the ϵ_ν and the λ_ν given by the author do not depend on the regular series-to-sequence method B. A necessary condition requires a representation of the ϵ_ν in terms of the coefficients of A: (1) $\epsilon_\nu = \sum_{n=\nu}^{\infty} \beta_n (a_{n\nu} - a_{n-1,\nu})$ with bounded β_n. For a normal

A, the "mean value theorems"

$$(2)\qquad \sum_{n=k+1}^{\infty}\left|\sum_{\nu=0}^{k}a_{n\nu}s_\nu-\sum_{\nu=0}^{k}a_{n-1,\nu}s_\nu\right|\leqq\sum_{n=0}^{k}|\sigma_n-\sigma_{n-1}|,$$

$$(3)\qquad \sum_{\nu=0}^{n}|a_{n\nu}s_\nu|\leqq\sum_{\nu=0}^{n}|\sigma_\nu-\sigma_{\nu-1}|,\quad \sigma_n=\sum_{\nu=0}^{n}a_{n\nu}s_\nu,$$

hold if $a_{n\nu}$ is monotone for $n\geqq\nu$ and $a_{n\nu}\to 0$ for $n\to\infty$. A lemma allows one to deduce convergence from the mere boundedness of the C transforms of each $|A|$-summable-to-zero sequence, provided A is normal and absolutely perfect. These means are used to show that (1) is sufficient for such A. There are similar theorems for the ϵ_ν. If $A=M(p_n)$ is the method of arithmetic means, (1) takes the elegant form $\sum_{\nu=0}^{n}\epsilon_\nu=O(1)$, $\epsilon_n=O(p_n/P_n)$. For the Cesàro means $A=C_\alpha$, $0<\alpha\leqq 1$, results similar to the Hardy-Bohr theorems and to a result by the reviewer [ibid. **51**, 85–91 (1948); these Rev. **10**, 31] are obtained. *G. G. Lorentz.*

Jurkat, Wolfgang. Über Rieszsche Mittel und verwandte Klassen von Matrixtransformationen. Math. Z. **57**, 353–394 (1953).

A triangular method of summation $A=(a_{n\nu})$ satisfies the mean value theorem $\mathfrak{M}$ (or $A\in\mathfrak{M}$) if

$$\left|\sum_{\nu=0}^{n}a_{n\nu}s_\nu\right|\leqq K\left|\sum_{\nu=0}^{n'}a_{m\nu}s_\nu\right|$$

for all s_ν, all $n, m=0, 1, \cdots$, and an appropriate $0\leqq n'\leqq n$ [see Jurkat and Peyerimhoff, Math. Z. **55**, 92–108 (1951); **56**, 152–178 (1952); these Rev. **13**, 934; **14**, 158]. This paper discusses products of methods satisfying $\mathfrak{M}$. For Riesz means, $R(\lambda_n,\kappa)\in\mathfrak{M}$ holds only if $0<\kappa\leqq 1$. Therefore, "general Cesàro means" $C^\delta(\omega)=C^\delta[s_\nu]$ of a sequence s_ν are introduced. If $0<\delta\leqq 1$,

$$C^\delta(\omega)=\Gamma(\delta+1)^{-1}\sum_{\nu=0}^{n}\{(\lambda_{n+1}-\lambda_\nu)^\delta-(\lambda_{n+1}-\lambda_{\nu+1})^\delta\}s_\nu,\qquad \lambda_n<\omega\leqq\lambda_{n+1},$$

and for an arbitrary κ,

$$C^\kappa(\omega)=(k+1)^{-1}\Gamma(\delta+1)^{-1}\sum_{\nu=0}^{n}\{(\lambda_{n+1}-\lambda_\nu)^\delta-(\lambda_{n+1}-\lambda_{\nu+1})^\delta\}t_\nu,$$

with $t_\nu=(\lambda_{\nu+k+1}-\lambda_\nu)C^k(\lambda_{\nu+1})/(\lambda_{\nu+1}-\lambda_\nu)$, $k=[\kappa]$. Under some conditions on the λ_n, $C^\kappa[s_\nu]/C^\kappa[1]\to s$ is equivalent to the $R(\lambda_n,\kappa)$ summability of s_ν to s. The author conjectures that if $C=AB$, then the convergence factors (c.f.) γ_ν of C are obtained from the c.f. ϵ_ν of A by means of $\gamma_\nu=\sum_{k=\nu}^{\infty}\epsilon_k b_{k\nu}$. This leads him to the determination of all c.f. of the Riesz means $R(\lambda_n,\kappa)$; for $0<\kappa\leqq 1$ the c.f. were given by the author [ibid. **54**, 262–271 (1951); these Rev. **13**, 340, 1139] and Peyerimhoff [ibid. **55**, 23–54 (1951); these Rev. **13**, 933]. Among other results we note limitations theorems which state that $\sigma_n=o(\eta_n)$ implies $\tau_n=o(\eta_n b_{nn}/a_{nn})$ where $\eta_n>0$ is increasing and σ_n, τ_n denote the AC and the BC transforms of a sequence s_ν, further inclusion theorems, some of which use the assumption that $\Delta^r A\in\mathfrak{M}$, where $\Delta^r A=(\Delta_\nu{}^r a_{n\nu})$.
G. G. Lorentz (Detroit, Mich.).

Zeller, Karl. Merkwürdigkeiten bei Matrixverfahren; Einfolgenverfahren. Arch. Math. **4**, 1–5 (1953).

Let $\mathfrak{A}, \mathfrak{B}, \cdots$ denote sets of sequences summable by matrix methods $A, B, \cdots$. The author shows that: (i) to each unbounded sequence d_n and each d there is a regular normal matrix D (ein Einfolgeverfahren) which sums exactly the sequences αd_n+c_n, c_n convergent, to $\alpha d+\lim c_n$; (ii) if A, C are regular, A row finite and $\mathfrak{A}\subset\mathfrak{C}$, there is a regular method B with $\mathfrak{A}\subset\mathfrak{B}\subseteq\mathfrak{C}$ which is consistent with A but not with C; (iii) there is a sequence of regular methods $F^{(\nu)}$, $\nu=1, 2, \cdots$, with $\mathfrak{F}^{(\nu+1)}\subset\mathfrak{F}^{(\nu)}$ (hence each $F^{(\nu)}$ sums divergent sequences) such that no divergent sequence is summable by all $F^{(\nu)}$. Darevsky [Izvestiya Akad. Nauk SSSR. Ser. Mat. **10**, 97–104 (1946); these Rev. **7**, 517] had slightly weaker results than (i), (ii), with D in (i) not necessarily normal and with $A=I$ in (ii). *G. G. Lorentz.*

Zeller, Karl. *FK*-Räume und Matrixtransformationen. Math. Z. **58**, 46–48 (1953).

The author rectifies a minor error in the proof of his Theorem 4.10c [Math. Z. **53**, 463–487 (1951); these Rev. **12**, 604]; he points out that Theorem 10, p. 47 in the book of S. Banach [Théorie des opérations linéaires, Warsaw, 1932] is inexact, Theorem 12, p. 51 incorrectly quoted, and corrects the proof of Theorem 13, p. 51. *G. G. Lorentz.*

Kadec, M. I. On a property of broken lines in n-dimensional space. Uspehi Matem. Nauk (N.S.) **8**, no. 1(53), 139–143 (1953). (Russian)

This note gives a new proof of Šklyarskiĭ's generalization [Uspehi Matem. Nauk **10**, 51–59 (1944); these Rev. **7**, 12] of Riemann's theorem on conditionally convergent series: The set of sums of rearrangements of a conditionally convergent series of vectors in n-space is a flat subspace of the space. This proof depends on the following result: For each integer n there is a number K_n such that for each set of N vectors r_j in Euclidean n-space, all of length $\leqq 1$ and such that $\sum_{j\leqq N}r_j=0$, the indices can be permuted so that for $1\leqq k\leqq N$ the vectors $\sum_{j\leqq k}r_j$ are all of length not greater than K_n. *M. M. Day* (Urbana, Ill.).

This proof is generally that of Steinitz [J. Reine Angew Math. 143, 128–175 (1913)]

Kochmański, T. Operations with multidimensional series. Arch. Méc. Appl., Gdańsk **3**, 293–345 (1951). (Polish. Russian summary)

Korenblyum, B. I. A general Tauberian theorem for the ratio of functions. Doklady Akad. Nauk SSSR (N.S.) **88**, 745–748 (1953). (Russian)

Let

$$f(x)=\int_0^\infty k(\xi/x)d\phi(\xi),\quad g(x)=\int_0^\infty k(\xi/x)d\psi(\xi)$$

and $f(x)/g(x)\to 1$ for $x\to\infty$. Then $\psi(x)/\phi(x)\to 1$. This Tauberian theorem is proved under the assumptions that $k(x)\geqq 0$, $k(0_+)>0$, $k(x)=o(x^{-\gamma})$ as $x\to\infty$ $(\gamma>0)$, $\int_0^\infty|k'|(1+x^\gamma)dx<\infty$, that the integral equation

$$\int_0^\infty k'(xt)s(t)dt=0,\quad 0<x<\infty,$$

has no non-trivial solutions $s(x)$ with $s(x)=O(x^\gamma)$, $x\to\infty$, and that ϕ, ψ are positive increasing with $\lim\phi(x)=\infty$, $\phi(y)/\phi(x)\leqq(y/x)^\gamma$ for large $x, y, x<y$. In case $k(x)=(1+x)^{-m}$, $0<\gamma<m$, this contains a theorem of M. V. Keldyš [Trudy Mat. Inst. Steklov. **38**, 77–86 (1951); these Rev. **13**, 738]. The proof, unlike that of Keldyš, is elementary and uses the inverse of l'Hôpital's theorem for the integrals of ϕ and ψ and the fact that each function $h(x)$ with $\|h\|=\int_0^\infty|h(x)|(1+x^\gamma)dx<+\infty$ is approximable, in this norm, by linear combinations of the form $\sum_{i=1}^{n}C_i k'(\alpha_i x)$, $\alpha_i>0$. *G. G. Lorentz* (Detroit, Mich.).

Fourier Series and Generalizations, Integral Transforms

Boas, R. P., Jr. Oscillation of partial sums of Fourier series. J. Analyse Math. **2**, 110–125 (1952). (Hebrew summary)

The author proves theorems which make precise the principle that when a function is smooth enough in a neighborhood of a point, the partial sums $s_n(x)$ of the Fourier series of the function oscillate around the value of the function at the point. Let $f(x)$ be real integrable of period 2π. Denote by $\bar{f}$ the conjugate function (assumptions are such that $\bar{f}$ exists); s_n and σ_n denote, respectively, the partial sums and the $(C, 1)$ means of the Fourier series of $f(x)$ for $x=0$; $E(x)$ denotes the even component of $f(x)$: $E(t)=f(t)+f(-t)-2f(0)$. Then $E(x)=0$ almost everywhere (and, therefore, $f(x)-f(0)$ is odd, and $s_n=f(0)$ for all n) if one of the following sets of conditions is satisfied. (A) $s_n \geqq f(0)$ $(n \geqq 0)$, $E(x)$ has in the neighborhood of 0 an absolutely continuous derivative, $x^{-1}E'(x)$ is integrable, and $E'(0)=0$. (B) $s_n \geqq f(0)$ $(n \geqq 0)$, in a neighborhood of 0, $\bar{f}(x)$ is absolutely continuous, and $x^{-1}\bar{f}'(x)$ is integrable. (C) $\sigma_n \geqq f(0)$ $(n \geqq 0)$, $\bar{f}''(0)=0$, $\bar{f}^{(\omega)}(0)$ exists, $\bar{f}'(x)$ is absolutely continuous in a neighborhood of 0, $x^{-2}\bar{f}''(x)$ is integrable. (D) $E(x)$ is constant in $(-\epsilon, \epsilon)$, $0<\epsilon<\pi$, either both $s_n-f(0)$ and $s_{n+1}-f(0)>-e^{-\delta n}$, or both $<e^{-\delta n}$, $\delta>0$, for a sequence of integers n of Pólya maximum density exceeding $(\pi-\epsilon)/\pi$. (E) In (D) δn can be replaced by $\theta(n)$, where $\theta(t)$ increases and $\int^{\infty} t^{-2}\theta(t)dt=\infty$, provided the sequence of integers has an ordinary density exceeding $(\pi-\epsilon)/\pi$. Theorems are proved in which f or $\bar{f}$ is supposed analytic in $[-\epsilon, \epsilon]$. Finally, one supposes only that f is differentiable in a neighborhood of 0. *S. Mandelbrojt.*

Boas, R. P., Jr. Integrability of trigonometric series. III. Quart. J. Math., Oxford Ser. (2) **3**, 217–221 (1952).

[Per le parti I e II vedi Duke Math. J. **18**, 787–793 (1951); Math. Z. **55**, 183–186 (1952); questi Rev. **13**, 549, 838.] Detta L la classe delle funzioni sommabili nell'intervallo $(0, r)$ l'autore prova i teoremi sequenti. (I) Se $f_n \downarrow 0$ e $b(x)=\sum_{n=1}^{\infty} f_n \sin nx$, allora, per $0 \leqq \gamma \leqq 1$, $x^{-\gamma}b(x) \in L$ se e solo se la serie $\sum_{n=1}^{\infty} n^{\gamma-1}f_n$ converge. (II) Se $f_n \downarrow 0$ e $a(x)=\sum_{n=1}^{\infty} f_n \cos nx$, allora, per $0<\gamma<1$, $x^{-\gamma}a(x) \in L$ se e solo se la serie $\sum_{n=1}^{\infty} f_n n^{\gamma-1}$ converge. (III) Se $f_n \downarrow 0$ per n abbastanza grande, mentre la serie $\sum_{n=1}^{\infty} f_n$ converge e si ha

$$a(x)=\tfrac{1}{2}f_0+\sum_{n=1}^{\infty} f_n \cos nx, \quad a(0)=0,$$

allora $a(x)/x \in L$ se e solo se la serie $\sum_{n=1}^{\infty} f_n \log n$ converge. (IV) Se la funzione $f(x)$, di cui $\sum_{n=1}^{\infty} b_n \sin nx$ è la serie di Fourier, non cambia segno in un intorno destro di 0 e se $b_n \geqq 0$, allora $f(x)/x \in L$ se e solo se converge la serie $\sum_{n=1}^{\infty} b_n$. *E. De Giorgi* (Roma).

Dzyadyk, V. K. On best approximation in the class of periodic functions having a bounded s-th derivative $(0<s<1)$. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 135–162 (1953). (Russian)

The author determines the best trigonometric approximation, in the C and L metrics, to a periodic function whose sth derivative $(0<s<1)$ is bounded or integrable and whose first Fourier coefficients vanish. In order to do this he first makes a detailed study of the best trigonometric approximation to the functions $\Psi_s(t)=\sum_1^{\infty} k^{-s}\cos(kt-s\pi/2)$. Finally he studies the trigonometric approximation of functions which have a singularity of the same kind as $\Psi_s(t)$. *R. P. Boas, Jr.* (Evanston, Ill.).

Timan, A. F. Approximation properties of linear methods of summation of Fourier series. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 99–134 (1953). (Russian)

The first part of the paper gives the proof of a theorem announced by the author in a previous note [Doklady Akad. Nauk SSSR (N.S.) **81**, 509–511 (1951); these Rev. **13**, 457]. The second part of the paper is devoted to the exploitation of the following idea. Let $\lambda_k^{(n)}$ $(k=0, 1, \cdots, n; n=0, 1, \cdots)$ be a triangular matrix satisfying $|\lambda_k^{(n)}| \leqq 1$, $\lambda_0^{(n)}=1$. With any Fourier series $\frac{1}{2}a_0+\sum a_k \cos kx+b_k \sin kx$ of a continuous function f we associate the polynomials

$$U_n(f; x; \lambda)=\frac{a_0}{2}\sum_{k=1}^{n}\lambda_k^{(n)}(a_k \cos kx+b_k \sin kx),$$

and for a class $\mathfrak{M}$ of functions f introduce the expression

$$\mathcal{E}_n(\mathfrak{M}, \lambda)=\sup_{x,\, f \in \mathfrak{M}} |f(x)-U_n(f; x; \lambda)|.$$

It is a well-known fact that the rapidity with which $\lambda_k^{(n)}$ tends to 1 for $k \to \infty$ and each n is reflected in the rapidity with which $\mathcal{E}_n$ tends to 0 as $n \to \infty$. The author investigates this connection when $\mathfrak{M}$ is the class $W^{(r)}H_\omega$ of functions having an rth derivative whose modulus of continuity does not exceed the given function $\omega(\delta)$, and class $\overline{W^{(r)}H_\omega}$ of functions conjugate to the functions in $W^{(r)}H_\omega$. He shows that both $\mathcal{E}_n(W^{(r)}H_\omega, \lambda)$ and $\mathcal{E}_n(\overline{W^{(r)}H_\omega}, \lambda)$ are not less than

$$\pi^{-2}\int_0^{\pi/2}\omega(2t/n)\sin t\, dt\left|\sum_{k=1}^{n}\frac{1-\lambda_k^{(n)}}{k^r(n-k+1)}-\frac{\log n}{n^r}\right| + O\left\{n^{-1}\omega(n^{-1})\sum_{k=1}^{n}\frac{1-\lambda_k^{(n)}}{k^r}\right\}+O(n^{-r}\omega(n^{-1}))$$

where the factor $O(1)$ on the right denotes quantities uniformly bounded for all matrices $\{\lambda_k^{(n)}\}$ considered. If in addition $\omega(t)$ is concave, the inequality holds even when the right side is multiplied by 2 and this is the best possible estimate. In the case when $\omega(t)=t^\alpha$, $0<\alpha<1$, the author obtains estimates from above for $\mathcal{E}_n(W^{(r)}H_\omega, \lambda)$ and $\mathcal{E}_n(\overline{W^{(r)}H_\omega}, \lambda)$. In the special case when $r=0$, $\omega(t)=t^\alpha$, $0<\alpha<1$, and when each row of the matrix $\lambda_k^{(n)}$ is a decreasing and concave sequence, the last two expressions are

$$2^{\alpha+1}\pi^{-\alpha}n^{-\alpha}\int_0^{\frac{1}{2}\pi} t^\alpha \sin t\, dt\sum_{k=1}^{n}\frac{\lambda_k^{(n)}}{n-k+1}+O(n^{-\alpha}).$$

A. Zygmund (Chicago, Ill.).

Bari, N. K. On primitive functions and trigonometric series converging almost everywhere. Mat. Sbornik N.S. **31(73)**, 687–702 (1952). (Russian)

Lusin a démontré [Mat. Sbornik **28**, 266–294 (1912)] qu'une condition nécessaire et suffisante pour qu'une fonction $f(x)$ admette une primitive (c'est-à-dire, une fonction $F(x)$ continue, avec $F'(x)=f(x)$ presque partout sur l'intervalle de définition de f) est que $f(x)$ soit mesurable et finie presque partout [Thèse, Moscou, 1916 = Mat. Sbornik **30**, 1–242 (1916)]. Menchoff a, d'autre part, démontré qu'à toute fonction mesurable et finie presque partout sur $[-\pi, \pi]$ correspond une série trigonometrique convergeant vers $f(x)$ presque partout sur $[-\pi, \pi]$ [Mat. Sbornik N.S. **9(51)**, 667–692 (1941); ces Rev. **3**, 106]. L'auteur du présent travail démontre un théorème contenant les deux résultats

cités. Voici le résultat fondamental obtenu: A toute fonction $f(x)$ mesurable et finie p.p. sur $I=[-\pi,\pi]$ correspond une fonction $F(x)$ continue sur I, telle que $F'(x)=f(x)$ p.p. sur I, et, en dérivant terme à terme la série de Fourier de $F(x)$, on obtient une série trigonometrique convergeant vers $f(x)$ p.p. Voici quelques lemmes qui servent. Lemme I. Soit $\phi(x)$ une fonction absolument continue sur $[-\pi,\pi]=I$, soit $\sigma>0$, et soit $(a,b)\subset I$. Il existe une fonction continue, à variation bornée sur I, $\chi(x)$, qui possède les propriétés suivantes: (a) $\chi(x)$ est constante sur tous les intervalles contigus à un ensemble parfait, de mesure nulle, $\bar{\omega}\subset(a,b)$, et $\phi(x)=\chi(x)$ pour x non-ϵ (a,b); (b) la variation totale de χ sur $[a,b]$ n'est pas supérieure à celle de ϕ sur le même intervalle; (c) on a $|\phi(x)-\chi(x)|\leqq\sigma$ sur I; (d) les coefficients de Fourier de $F(x)=\phi(x)-\chi(x)$ sont de l'ordre $o(1/n)$. Lemme II. Soit $[a,b]\subset[-\pi,\pi]=I$, et soit $F(x)$ une fonction continue à variation bornée sur I, égale à zéro lorsque x non-ϵ $[a,b]$. En désignant par $S_n(x)$ la somme de Fourier, partielle, de F, et par V la variation totale de F sur $[a,b]$, on a $|S_n'(x)|\leqq V/2\rho$, où ρ est la plus petite distance de x à l'intervalle $[a,b]$ (x non-ϵ $[a,b]$). Voici, enfin, un corollaire d'un théorème de Menchoff: A toute fonction $f(x)$ mesurable et finie p.p. sur I correspond une suite d'ensembles $\{P_n\}$, parfaits, denses nulle part, $P_n\cap P_m=0$ $(n\neq m)$, $m[\bigcup P_n]=2\pi$; sur P_n $(n\geqq 1)$ on a $f(x)=f_p(x)$, où $f_p(x)$ est une fonction continue dont la série de Fourier converge uniformément sur I.

S. Mandelbrojt (Houston, Texas).

Tsuchikura, Tamotsu. Remark on a theorem of Erdös and a problem of Zalcwasser. J. Math. Tokyo **1**, 27–31 (1951).

The results of this note are related to a paper of Zalcwasser [Studia Math. **6**, 82–88 (1936)] and a recent paper of Erdös [Acta Sci. Math. Szeged **12**, Pars A, 11–17 (1950); these Rev. **12**, 164]. All estimates in the following hold for almost all x. Let $f(x)$ be bounded and measurable, $\bar{f}(x)$ its conjugate (if it exists), $s_m(x)$ and $\bar{s}_m(x)$ the partial sums of the respective Fourier series. Let $\{p_k\}$ be an increasing sequence of integers. Then

$$n^{-1}\sum_{k=1}^{n}|s_{p_k}(x)|=O(\log\log n),\quad n^{-1}\sum_{k=1}^{n}|\bar{s}_{p_k}(x)|=O(\log\log n);$$

$$s_{p_k}(x)=O(\log k),\quad \bar{s}_{p_k}(x)=O(\log k).$$

If $f(x)$ is continuous, O can be replaced by o. Now let $f(x)$ have the modulus of continuity $o(1/\log\log(1/h))$. Then

$$n^{-1}\sum_{k=1}^{n}|s_{p_k}(x)-f(x)|\to 0,\quad n^{-1}\sum_{k=1}^{n}|\bar{s}_{p_k}(x)-f(x)|\to 0.$$

Finally, let $L_n(f;x)$ denote the Lagrange polynomial of the continuous function $f(x)$ taken at the zeros of Tchebychev polynomials. Then

$$n^{-1}\sum_{k=1}^{n}|L_{p_k}(f;x)|=o(\log\log n),\quad L_{p_k}(f;x)=o(\log k),$$

and if $f(x)$ has the modulus mentioned above,

$$n^{-1}\sum_{k=1}^{n}|L_{p_k}(f;x)-f(x)|\to 0.$$

G. Szegö (Stanford, Calif.).

Lalaguë, Pierre. Classes de fonctions indéfiniment dérivables sommes de séries de Fourier lacunaires. C. R. Acad. Sci. Paris **236**, 887–889 (1953).

Let $E=\{\lambda_n\}$ be an increasing sequence of positive integers, and let $\{M_n\}$ be a sequence of positive numbers. An infinitely differentiable and periodic function $f(x)$ on the line with period 2π is said to belong to the class $C^E\{M_n\}$ if: (i) $|f^{(n)}(x)|\leqq A^nM_n$ for some constant A;

(ii) $$f(x)=a_0+\sum(a_n\cos\lambda_nx+b_n\sin\lambda_nx).$$

The author extends to the class $C^E\{M_n\}$ theorems of inclusion, derivability, analyticity, and quasi-analyticity established by Mandelbrojt in the case $\lambda_n=n$ [Séries de Fourier . . . , Gauthier-Villars, Paris, 1935; Rice Inst. Pamphlet **29**, no. 1 (1942); these Rev. **3**, 292]. A certain convex regularisation of $\log M_n$ depending on the sequence E renders here the same service as the ordinary convex regularisation in the case of Mandelbrojt. *S. Agmon.*

Izumi, Shin-ichi. Notes on Fourier analysis. XVI. On the strong law of large numbers and gap series. Tôhoku Math. J. (2) **3**, 89–103 (1951).

In the first part of this paper the author proves, among other results, that if $\int_0^1 f(x)dx=0$ and

$$(*)\quad \left(\int_0^1[f(x+t)-f(x)]^2dx\right)^{\frac{1}{2}}=O\left[\left(\log\log\frac{1}{t}\right)^{-\frac{1}{2}-\epsilon}\right],$$

the series $\sum_1^\infty f(n_kx)/k$ converges almost everywhere. Here $\{n_k\}$ is a gap sequence such that $n_{k+1}/n_k>q>1$.

In the second part condition (*) is replaced by a stronger condition, i.e., $(\log\log t^{-1})^{-\frac{1}{2}-\epsilon}$ is replaced by t^α, $0<\alpha\leqq 1$. The principal result is now that $\sum c_k^2(\log\log k)^2<\infty$ implies a.e. convergence of $\sum_1^\infty c_kf(n_kx)$. Finally, if in addition to the above strengthening of (*) one also assumes $\sum_1^\infty(n_{k+1}/n_k)^{-2\alpha}<\infty$, then $\sum_1^\infty c_k^2<\infty$ implies convergence a.e. of $\sum_1^\infty c_kf(n_kx)$. This last result is a generalization of a result of the reviewer [Bull. Amer. Math. Soc. **55**, 641–665 (1949), in particular, pp. 652–654; these Rev. **11**, 161].

M. Kac (Ithaca, N. Y.).

Kreĭn, M. G. On a generalization of investigations of Stieltjes. Doklady Akad. Nauk SSSR (N.S.) **87**, 881–884 (1952). (Russian)

The author continues to issue highly concentrated progress summaries on the analytical theory of a vibrating loaded stretched string. The main novelty in this paper appears to be the study of "spectral functions" associated with a mass-distribution $\sigma(x)$. Let $\sigma(x)$ $(0\leqq x<l;\ 0<l\leqq\infty)$ be non-decreasing, with $\sigma(x)=\sigma(x-0)$, $\sigma(0)=0$, and let $\varphi(x;\lambda)$ be given by $\varphi(x;\lambda)=1-\lambda\int_0^x(x-s)\varphi(s;\lambda)d\sigma(s)$. Let L_σ be the Hilbert space of all σ-measurable functions $f(x)$ $(0\leqq x<l)$ with σ-integrable square, L_σ^0 the sub-space for which $f(x)$ vanishes throughout a left neighbourhood of $x=l$. A non-decreasing function $\tau(\lambda)$ $(-\infty<\lambda<\infty;\ \tau(0)=0)$ is called a spectral function of σ if the mapping

$$F(\lambda)=\int_0^l f(x)\varphi(x;\lambda)d\sigma(x)$$

maps L_σ^0 isometrically into the Hilbert space L_τ of τ-measurable functions $F(\lambda)$ $(-\infty<\lambda<\infty)$ with τ-integrable square. Theorem 1 gives four sets of conditions, each necessary and sufficient, that to given σ there should correspond at least two distinct τ; one such set is $\sigma(l)=\lim_{x\to l-0}\sigma(x)<\infty$, $\int_0^l x^2d\sigma(x)<\infty$. He then concentrates on "positive" spectral functions, for which $\tau(\lambda)=0$ for $\lambda<0$; Theorem 2 gives explicit formulae involving these functions. The connection with Stieltjes continued fraction theory and moment problems is then discussed, and finally (Theorems 3 and 4) inverse problems, giving necessary and sufficient conditions for a σ to exist for given τ. There appears here a connection with his previous note [same Doklady (N.S.) **82**, 669–672 (1952); these Rev. **14**, 649] on the "transition function"

and time of passage of a wave; however, the dynamical aspects are on this occasion expressly excluded for lack of space. *F. V. Atkinson* (Ibadan).

Agmon, Shmuel. Complex variable Tauberians. Trans. Amer. Math. Soc. **74**, 444–481 (1953).

Das Laplace-Integral (1) $\int_0^\infty e^{-su}da(u)$ $(s=\sigma+it)$ sei für $\sigma>0$ konvergent und stelle dort die Funktion $f(s)$ dar. Ziel der Arbeit ist es, unter sehr allgemeinen Annahmen über die Funktion $a(u)$ und das Verhalten von $f(s)$ in der Nähe von $s=0$ zu schliessen auf das Verhalten von $a(u)$ oder allgemeiner von $a(u+h)-a(u)$ für $u\to\infty$. Verf. erhält so Verallgemeinerungen des Satzes von Ikehara [J. Math. Physics **10**, 1–12 (1931)]. Beim Beweis stützt sich Verf. auf die von ihm schon früher [Ann. Sci. Ecole Norm. Sup. (3) **66**, 263–310 (1949); diese Rev. **11**, 427] zum Zwecke der Untersuchung von Singularitäten bei Dirichlet-Reihen verwandte "Methode der Hauptindizes". In §§2–3 werden Hilfsmittel bereitgestellt. (1) konvergiere in $\sigma>0$ absolut, und es sei $c_n=\int_n^{n+1}|da(u)|$ gesetzt. $\{q_n\}$ sei eine gewissen Regularitätsbedingungen genügende (sog. "reguläre") Majorante von $\{c_n\}$. Dann ist die Familie F der Funktionen $f_u(s)=[f(s)-\int_0^u e^{-su}da(u)]/q_{[u]}e^{-su}$ in jedem abgeschlossenen Teil des Regularitätsgebiets von $f(s)$ gleichmässig beschränkt. Ist $f(s)$ über $\sigma=0$ hinaus fortsetzbar, so sind sämtliche Grenzfunktionen von F in $\sigma\geqq 0$, sowie in den Regularitätspunkten von $f(s)$ auf $\sigma=0$ regulär; jede Grenzfunktion gestattet überdies eine explizite Darstellung. Wichtig ist, wann nicht alle Grenzfunktionen $\equiv 0$ sind: Sei $\{x_k\}$ eine Folge von Hauptindizes [d.h., die obengenannte Folge $\{q_n\}$ erfüllt auch noch $q_n\leqq Mc_n$ für $n=x_k$ (M fest)], und $a(u)$ gehöre zur Klasse Δ [d.h., es gibt positive Zahlen d und $\gamma<\pi$, sodass für jedes $u_0>0$ ein Winkelraum Σ_{u_0} mit Spitze in 0 und Öffnung $<\gamma$ existiert, in dem $a(u_2)-a(u_1)$ liegt für $u_1<u_2$ und $u_1, u_2\in\langle u_0-d, u_0+d\rangle$]. Dann hat jede Grenzfunktion der Folge $\{f_{x_k}(s)\}$ mindestens eine Singularität auf $\sigma=0$. Die §§4–7 bringen Anwendungen dieser Sätze. (1) konvergiere in $\sigma>0$ absolut, und es sei $a(u)\in\Omega$ [Funktionsklasse $\subset\Delta$, enthält z.B. alle monoton wachsenden Funktionen $a(u)$]. Eine zugeordnete Taylor-Dirichlet-Reihe (2) $\phi(s)=\sum_{n=0}^\infty a_n e^{-ns}$ konvergiere in $\sigma>0$, und $f(s)-\phi(s)$ sei in $s=0$ regulär. Dann ist

$$a(u)=\sum_{n<u}a_n+c+O(q_{[u]})$$

und insbesondere (3) $\int_u^{u+1}|da(t)|=O(q_{[u]})$, wobei c eine Konstante ist und $\{q_n\}$ eine "reguläre" Majorante von $\{a_n\}$ bedeutet. Die einseitige Abschätzung (3) kann unter zusätzlichen Annahmen über $\{a_n\}$ in eine zweiseitige verschärft werden; weiter wird der allgemeinere Fall $a(u)\in\Delta$ behandelt. Schliesslich wird auf das asymptotische Verhalten von $a(u+h)-a(u)$ geschlossen. In (1) sei $a(t)\in\Delta$ und $f(s)$ sei für ein $T>0$ und jedes $\epsilon>0$ in den Halbstreifen $0<\epsilon\leqq|t|\leqq T$, $\sigma>0$ beschränkt. Es existiere eine Funktion (2) mit $\liminf|a_n|>0$, und so dass $\log|a_n|$ eine konkave Funktion von n ist. Weiter sei $\phi(s)$ auf $\sigma=0$ regulär mit Ausnahme der Punkte $s=2k\pi i$ $(k=0,\pm1,\cdots)$, und $f(s)-\phi(s)$ sei in einem Halbstreifen $|t|\leqq T_0$, $\sigma>0$, beschränkt. Dann gilt $a(u+h)-a(u)\sim ha_{[u]}$ $(u\to\infty)$, gleichmässig für $0<h_1\leqq h\leqq h_2$. Für $a_n=1$ in (2) erhält man ein den Ikeharaschen Satz verallgemeinerndes Ergebnis. Als Anwendung wird abschliessend für die Summe der reziproken Primzahlen bewiesen:

$$\sum_{p<u}p^{-1}=\log\log u+c_0+o(1/\log u)\quad (c_0=\text{Konstante}).$$

D. Gaier (Stuttgart).

Scott, E. J. Jacobi transforms. Quart. J. Math., Oxford Ser. (2) **4**, 36–40 (1953).

The finite Jacobi transform of $F(x)$ is defined to be $J\{F(x)\}=f^{(\alpha,\beta)}(n)=\int_{-1}^{+1}F(x)P_n^{(\alpha,\beta)}(x)dx$ where $P_n^{(\alpha,\beta)}(x)$ is the nth Jacobi polynomial. The orthogonality properties of the Jacobi polynomials yield the inverse transform $F(x)=\sum_{n=0}^\infty b_n^{-1}f_{(n)}^{(\alpha,\beta)}(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)$. The Jacobi transform is a generalization of the finite Legendre transform [Tranter, same J. (2) **1**, 1–8 (1950); these Rev. **12**, 25]. It is shown that an application of the Jacobi transform excludes the group of terms

$$\partial/\partial x\{(1-x^2)\partial U(x,t)/\partial x\}+[(\alpha-\beta)+(\alpha+\beta)x]\partial U(x,t)/\partial x$$

from a differential equation, and an application of this is made to the determination of the temperature $U(x,t)$ of a non-homogeneous one-dimensional bar with ends at $x=+1$ and $x=-1$ and with thermal conductivity $K(1-x^2)$ and a continuous heat source of strength $(ax+b)\partial U(x,t)/\partial x$. The analysis is purely formal. *F. Goodspeed.*

Langebartel, Ray G. The convolutions of the kernel $x^\lambda-t^\lambda$. J. Indian Math. Soc. (N.S.) **15** (1951), 118–128 (1952).

Dato il nucleo

$$h(x,y)=\begin{cases}x^\lambda-y^\lambda & \text{per } 0\leqq y\leqq x\\ 0 & \text{per } y<0 \text{ ovvero } y>x,\end{cases}$$

se ne studiano i nuclei iterati $h^\alpha(x,y)$ d'ordine α reale >0, facendo uso del metodo dei nuclei associati [D. G. Bourgin, Revista Ci., Lima **51**, nos. 3–4, 5–46 (1949); questi Rev. **13**, 127], ovvero, più rapidamente ma solo per α intero, facendo uso di formole di derivazione od integrazione d'ordine non intero già usate da A. Erdélyi [cfr. Quart. J. Math., Oxford Ser. **10**, 176–189 (1939); questi Rev. **1**, 117]. Secondo Bourgin si dice associato ad $h(x,y)$ un nucleo $L(p,t)$ per cui sia (1) $\int_a^\infty L(p,t)h(t,x)dt=p^{-1}L(p,x)$; si dimostra allora che i nuclei iterati d'ordine intero n verificano la

$$\int_a^\infty L(p,t)h^n(t,x)dt=p^{-n}L(p,x),$$

e si cerca di definire i nuclei iterati d'ordine reale $\alpha>0$ in modo da verificare la (2) $\int_a^\infty L(p,t)h^\alpha(t,x)dt=p^{-\alpha}L(p,x)$. Ora, l'equazione integrale (1) nella incognita $L(p,t)$ è agevolmente trasformata in una equazione differenziale del tipo di Bessel; l'equazione integrale (2) nella incognita $h^\alpha(t,x)$ è studiata da C. S. Meijer [Nederl. Akad. Wetensch., Proc. **43**, 599–608, 702–711 (1940); questi Rev. **2**, 96] che ne esprime la soluzione mediante un integrale curvilineo nel campo complesso; ciò permette in definitiva di esprimere $h^\alpha(x,y)$ mediante funzioni ipergeometriche: si verifica poi a posteriori che è $h^{\alpha+\beta}(x,y)=\int_y^x h^\alpha(x,t)h^\beta(t,y)dt$ per $\alpha,\beta>0$. Col metodo di Erdélyi si ottengono le stesse espressioni per $h^n(x,y)$ $(n=1,2,3,\cdots)$. *F. Bertolini* (Roma).

Fox, Charles. Iterated transforms. Canadian J. Math. **4**, 149–161 (1952).

È ben noto che sotto opportune ipotesi, iterando una trasformazione di Laplace si ottiene una trasformazione di Stieltjes, ossia che il sistema d'eguaglianze

$$f(x)=\int_0^{+\infty}e^{-xt}g(t)dt,\quad g(x)=\int_0^{+\infty}e^{-xt}h(t)dt$$

implica l'eguaglianza

$$f(x)=\int_0^{+\infty}\frac{h(t)}{x+t}dt.$$

Nella presente nota si ottengono varie generalizzazioni di questo enunciato, delle quali la più comprensiva è la seguente. Siano (1) $M(s)$, $N(s)$ due funzioni della variabile complessa s, limitate sulla retta $s=\frac{1}{2}+i\tau$ (τ reale) e verificanti la condizione $M(s)N(1-s)\equiv 1$; (2) $p(x)$, $q(x)$, $h(x)$ tre funzioni della variabile reale x, della classe $L^2(0,+\infty)$; (3) $P(s)=\int_0^\infty p(u)u^{s-1}du$, $Q(s)=\int_0^{+\infty}q(u)u^{s-1}du$ le trasformate di Mellin (supposte limitate sulla retta $s=\frac{1}{2}+i\tau$) di $p(x)$, $q(x)$ rispettivamente;

$$(4)\qquad \mathcal{P}(x)=\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}P(s)x^{-s}M(1-s)ds,$$

$$\mathcal{Q}(x)=\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}Q(s)x^{-s}N(1-s)ds;$$

$$(5)\qquad \mathcal{K}(x)=\int_0^{+\infty}p(xt)q(t)dt.$$

Con queste posizioni, il sistema di eguaglianze

$$f(u)=\int_0^{+\infty}\mathcal{Q}(ut)g(t)dt,\quad g(u)=\int_0^{+\infty}\mathcal{P}(ut)h(t)dt$$

implica l'eguaglianza

$$f(u)=\int_0^{+\infty}\mathcal{K}(t)h(ut)dt.$$

Nell'ulteriore ipotesi $M(s)\equiv N(s)$, si ha inoltre

$$\int_0^{+\infty}f(ut)h(t)dt=\int_0^{+\infty}g(ut)g(t)dt.$$

Si noti che: (a) se $M(s)$ ed $N(s)$ sono della classe $L^2(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty)$, allora le funzioni

$$m(u)=\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}M(s)u^{-s}ds,\quad n(u)=\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}N(s)u^{-s}ds$$

sono della classe $L^2(0,+\infty)$, e la (4) si scrive

$$(4')\quad \mathcal{P}(x)=\int_0^{+\infty}p(xt)m(t)dt,\quad \mathcal{Q}(x)=\int_0^{+\infty}q(xt)n(t)dt;$$

(b) posto $q(x)\equiv p(x)\equiv e^{-x}$, se la ipotesi (a) è verificata le $\mathcal{P}(x)$ e $\mathcal{Q}(x)$ non sono che le trasformate di Laplace di $m(u)$, $n(u)$ rispettivamente, se la ipotesi (a) non è verificata le (4) permettono di estendere il formalismo della trasformazione di Laplace anche a casi in cui la (4') non avrebbe senso, senza introdurre funzioni "impulsive" o comunque generalizzate; (c) posto $M(s)\equiv N(s)\equiv 1$ abbiamo $\mathcal{P}(x)\equiv p(x)$, $\mathcal{Q}(x)\equiv q(x)$, e nella ipotesi $p(x)\equiv q(x)\equiv e^{-x}$ si ritrova la proprietà enunciata al principio.

F. Bertolini (Roma).

Chakrabarty, N. K. Sur le calcul symbolique à deux variables. Ann. Soc. Sci. Bruxelles. Sér. I. **67**, 23–27 (1953).

Let $\phi(p,q)$ denote the transform or image of the function $f(x,y)$ of two variables in the sense

$$\phi(p,q)=pq\int_0^\infty\int_0^\infty e^{-px-qy}f(x,y)dxdy.$$

Then $f(x^2,y^2)$ has the transform

$$pq\pi^{-1}\int_0^\infty\int_0^\infty\exp\left(-\tfrac{1}{4}p^2s^2-\tfrac{1}{4}q^2t^2\right)\phi(s^{-2},t^{-2})dsdt.$$

If $\psi(u)$ denotes the corresponding one-dimensional transform of $g(x)$, then

$$\int_0^\infty\frac{\phi(p,q)g(p)}{pq}dp=\int_0^\infty\int_0^\infty\frac{f(x,y)\psi(x)}{x}e^{-xy}dxdy.$$

These two new formulas, together with the convolution property of the two-dimensional transformation, are used to obtain transforms of certain Bessel and hypergeometric functions. *R. V. Churchill* (Ann Arbor, Mich.).

Thosar, Yeshwant V. Two theorems on operational calculus. Math. Z. **57**, 336–348 (1953).

Writing $f(p)\doteqdot h(x)$ when $f(p)=p\int_0^\infty e^{-px}h(x)dx$, the author uses results by van der Pol and Niessen to show:

(i) If $f(p)\doteqdot h(x)$ and $f(p^{1/2})\doteqdot x^{1/2}F(1/x)$, then $F(p)\doteqdot h(2x^{1/2})/(\pi x)^{1/2}$.

(ii) If $f(p)\doteqdot h(x)$ and $(2/p)F(p^2/4)\doteqdot h(x^2)$, then $F(p)\doteqdot f(1/x)/(\pi x)^{1/2}$.

He applies these rules formally to a large number of examples. Not all of these are correct, and hardly any of the correct ones are new. *A. Erdélyi* (Pasadena, Calif.).

Kumar, Ram. A pair of functions which are Hankel transforms of each other, Ganita **3**, 79–84 (1952).

The author computes the Hankel transform of order ν of $x^{\nu-3/2}E(\alpha,\beta::\frac{1}{2}x^2)$ where E is MacRobert's function.

A. Erdélyi (Pasadena, Calif.).

Jaiswal, J. P. Two properties of Meijer transform. Ganita **3**, 85–90 (1952).

Continuing earlier papers [Math. Z. **55**, 385–398 (1952); Ann. Soc. Sci. Bruxelles. Sér. I. **66**, 55–60, 131–151 (1952); these Rev. **14**, 43, 640] the author computes the effect of a Meijer transform followed by a Laplace transformation, and the Meijer transform of a function which is self-reciprocal in the Hankel transformation. *A. Erdélyi.*

Poli, L. Calcul symbolique et équations aux dérivées partielles. Cahiers Rhodaniens **4**, 13–27 (1952).

An expository treatment of operational calculus based on Laplace transformations in one and two dimensions, with examples in integral and differential equations.

R. V. Churchill (Ann Arbor, Mich.).

Kinokuniya, Yoshio. On operational calculus. Mem. Muroran Coll. Tech. **1**, no. 2, 1–13 (1951).

Kinokuniya, Yoshio. On operational equations. Mem. Muroran Coll. Tech. **1**, no. 1, 13–24 (1950).

In these two papers the author gives a more or less formal discussion of operators acting on functions (having in mind mainly differential operators). The papers are not written very lucidly, it is difficult to say what the author is driving at, and even more difficult to know whether he has any result of interest. *A. Erdélyi* (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Chamberlin, Eliot, and Wolfe, James. The critical points of certain polynomials. Duke Math. J. **20**, 71–76 (1953).

The authors extend the results of Walsh [The location of critical points of analytic and harmonic functions, Amer. Math. Soc. Colloq. Publ., v. 34, New York, 1950, pp. 32–33;

these Rev. 12, 249] concerning the critical points of polynomials of the form $p(x)=(x^2+1)^k(x-a_1)\cdots(x-a_m)$ where the a_i are all real. They develop necessary and sufficient conditions on δ and γ so that, if $\delta\leq a_i\leq\gamma$ for all i, then $p(x)$ has non-real critical points $z\neq\pm i$. Their method involves establishing the existence of polynomials

$$p(x)=(x^2+1)^k(x-A)^r(x-B)^{n-r}$$

having a triple critical point in the interval $A<x<B$.

M. Marden (Milwaukee, Wis.).

Dimić, Platon. Une remarque sur les erreurs des valeurs évaluées par la formule d'interpolation de Lagrange. Bull. Soc. Math. Phys. Macédoine **3**, 29–34 (1952). (Serbo-Croatian. French summary)

The author asks how much the Lagrange interpolation polynomial of degree n can be changed by changes of at most a given amount in the data, and answers the question for $n=1, 2$. *R. P. Boas, Jr.* (Evanston, Ill.).

Geronimus, Ya. L. On mean and uniform approximations. Doklady Akad. Nauk SSSR (N.S.) **88**, 597–599 (1953). (Russian)

The author states several theorems with indications of proofs. (1) A sufficient condition, involving the degree of approximation by polynomials in pth mean to a function $\psi(\theta)$, for the analytic function with boundary values $\psi(\theta)$ on $|z|=1$ to be continuous in the closed unit circle. (2) If $\omega_p(x)$ denotes the integral modulus of continuity of index p of a given $f(\theta)$ of L^p, the condition $\int_0^h y^{-1-1/p}\omega_p(y)dy<\infty$ is sufficient for $f(\theta)$ to be equivalent to a continuous function whose ordinary modulus of continuity has an upper bound determined by $\omega_p(x)$. (3) Refinement of (1) and (2) when the given function is almost everywhere bounded and bounded away from zero. (4) An asymptotic formula in the closed unit circle for the orthogonal polynomials (on the unit circle) with weight $p(\theta)$ satisfying the conditions of (2) and (3) (index $p=2$). *R. P. Boas, Jr.*

Makai, E. On systems of polynomials orthogonal in two intervals. Publ. Math. Debrecen **2**, 222–228 (1952).

Let $\rho_1(x)$ and $\rho_2(x)$ be two given non-negative weights over the disjoint real intervals I_1 and I_2, respectively. Generalizing a classical problem dealt with by Heine and Stieltjes, the author proves the existence and unicity of a set of polynomials $p_k(x)$, $0\leq k\leq n$, of degree n satisfying the conditions

$$\int_{I_1}p_i(x)p_k(x)\rho_1(x)dx=\int_{I_2}p_i(x)p_k(x)\rho_2(x)dx=0,\quad i\neq k,$$

provided all moments involved in these conditions exist. It is further shown that the exact degree of $p_k(x)$ is n, and moreover that, in a suitable numbering of these polynomials, $p_k(x)$ has k zeros in I_1 and $n-k$ zeros in I_2. *G. Szegő.*

Weber, Maria, and Erdélyi, A. On the finite difference analogue of Rodrigues' formula. Amer. Math. Monthly **59**, 163–168 (1952).

Gli autori studiano i sistemi di polinomi ortogonali che si possono ottenere dalla seguente formula a differenze finite:

$$P_n(x)=\frac{1}{\rho(x)}\Delta^n[\rho(x-n)X(n)X(x-1)\cdots X(x-n+1)],\qquad n=0, 1, \cdots,$$

ove $\rho(x)$ è una funzione qualsiasi e $X(x)$ un polinomio. Affinchè $P_n(x)$ sia un polinomio di grado n per $n=0, 1, \cdots$ occorre che $X(x)$ sia di grado ≤ 2 e $\rho(x)$ una combinazione di funzioni euleriane; i sistemi di polinomi così ottenuti possono essere numerati.

Analoghi risultati sono stati ottenuti da W. Hahn per il corrispondente problema relativo alle differenze quozienti [Math. Nachr. **2**, 4–34, 263–278, 340–379 (1949); questi Rev. **11**, 29, 356, 720] e precedentemente da F. Tricomi nel caso delle derivate ordinarie [Serie ortogonali di funzioni, S. I. E. Istituto Editoriale Gheroni, Torino, 1948; questi Rev. **10**, 188]. *C. Pucci* (Roma).

Parodi, Maurice. Sur une méthode d'étude des polynomes récurrents. C. R. Acad. Sci. Paris **236**, 1626–1628 (1953).

Consider a sequence of polynomials $y_n(x)$, $n=0, 1, \cdots$, defined by the recurrence relation

$$f_1(n)y_n(x)+[f_2(n)-f_3(n)x]y_{n-1}(x)+f_4(n)y_{n-2}(x)=0,\qquad y_0=1,\quad y_{-1}=0,$$

where the functions $f_k(n)$ are real and $f_3(n)$ has no positive integral zeros. This recurrence relation leads to a system of linear equations in the unknowns $y_1(x), \cdots, y_n(x)$, and the zeros of the polynomial $y_n(x)$ are the characteristic values of a certain matrix. From known properties of matrices, a number of results are stated concerning the zeros of the polynomial $y_n(x)$. *E. Frank* (Chicago, Ill.).

Special Functions

Vadnal, Alojzij. Quelques propriétés du double logarithme et la somme des séries du type:

$$\sum_{n=1}^{\infty}\frac{n^p}{n!}.$$

Bull. Soc. Math. Phys. Serbie **4**, no. 3–4, 11–15 (1952). (Slovenian. Serbo-Croatian and French summaries)

Kreyszig, E. Der allgemeine Integralkosinus $\mathrm{Ci}(z, \alpha)$. Acta Math. **89**, 107–131 (1953).

In an earlier paper [Acta Math. **85**, 117–181 (1951); these Rev. **12**, 825] the author discussed the generalized sine integral. In the present paper he gives a similar discussion of

$$\mathrm{Ci}(z, \mu)=\int_0^z t^{-\mu}\cos t\,dt,\quad \mathrm{Re}\,\mu<1.$$

The paper concludes with 5 pages of numerical tables and 4 pages of diagrams. *A. Erdélyi* (Pasadena, Calif.).

Silverman, Louis L. Generalization of hyperbolic and trigonometric functions. Riveon Lematematika **6**, 53–60 (1953). (Hebrew. English summary)

For k, n integers, and $0\leq k\leq n$, let $h_k(x; n)$ be the sum of those terms in the exponential series whose index is congruent to k modulo n. Clearly $h_0(x; 2)$ and $h_1(x; 2)$ are, respectively, $\cosh x$ and $\sinh x$. The author investigates $h_k(x; 3)$, $k=0, 1, 2$, and also the similar series corresponding to trigonometric (rather than hyperbolic) functions. He gives no references, and seems unaware of the fact that the subject has been treated often and at some length. [See, for instance, L. Poli, Cahiers Rhodaniens **1** (1949) [these Rev. **12**, 178] where a bibliography of 26 items is given.]

A. Erdélyi (Pasadena, Calif.).

Jaffé, H. H. Some overlap integrals involving d orbitals. J. Chem. Phys. **21**, 258–263 (1953).

"Formulas are presented for overlap integrals for overlap of $d\pi$ with $p\pi$ or $d\pi$ orbitals, and for $d\delta$ with $d\delta$ orbitals. Slater AO's were used throughout. Numerical tables for these integrals for a variety of values of the parameters ρ and τ are given." (Author's summary.) The integrals occurring in this work are related to incomplete gamma functions. *A. Erdélyi* (Pasadena, Calif.).

Brafman, Fred. A set of generating functions for Bessel polynomials. Proc. Amer. Math. Soc. **4**, 275–277 (1953).

The following (divergent) generating function is established:

$$ {}_2F_0(a, c-a; R)\,{}_2F_0(a, c-a; S) \sim \sum_{n=0}^{\infty} \frac{(a)_n (c-a)_n}{n!} t^n\, {}_2F_0(-n, c+n; x), $$

$$ 2R = t + (t^2-4xt)^{1/2}, \quad 2S = t - (t^2-4xt)^{1/2}. $$

The symbol $\sim$ signifies that formal multiplication of the two power series on the left yields the same coefficient of t^n as occurs on the right. *G. Szegö* (Stanford, Calif.).

Dörr, Johannes. Untersuchung einiger Integrale mit Bessel-Funktionen, die für die Elastizitätstheorie von Bedeutung sind. Z. Angew. Math. Physik **4**, 122–127 (1953).

To facilitate the numerical computation of the integrals

$$ \int_0^\infty J_m^2(t)\frac{dt}{t+a}, \qquad \int_0^\infty J_m^2(t)\frac{dt}{t-2}, $$

$$ \int_0^\infty J_m^2(t)\frac{dt}{(t+a)^2}, \qquad \int_0^\infty J_m^2(t)\frac{dt}{(t-t_{m,n})^2}, $$

where m is an integer, a is positive, $t_{m,n}$ is a positive zero of the Bessel function $J_m(t)$, and the second integral is a Cauchy principal value, the author uses an integral representation of $J_m^2(t)$ and transforms each of the first two integrals into a sum whose first term is a product of Bessel functions, and the second term a finite integral of Struve's function of order zero multiplied by a trigonometric function. Corresponding representations of the last two integrals are obtained by differentiation with respect to a.

A. Erdélyi (Pasadena, Calif.).

Vacca, Maria Teresa. Sulle derivate delle funzioni di Bessel rispetto all'ordine, nel caso in cui questo è la metà di un intero dispari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 229–233 (1952).

Continuing Tricomi's investigations [same Rend. (8) **12**, 227–233 (1952); these Rev. **14**, 270] the author obtains an expression for $\dot{Y}(0, 0; x)$ and derives expressions for the partial derivatives of Bessel functions with respect to the order. When the order is half of an odd integer, these partial derivatives may be expressed in terms of the exponential integral and related functions. *A. Erdélyi.*

Makai, E. On a monotonic property of certain Sturm-Liouville functions. Acta Math. Acad. Sci. Hungar. **3**, 165–172 (1952). (Russian summary)

Let $x_{\nu k}$, $k=1, 2, 3, \cdots$, denote the non-negative zeros of $J_\nu(x)$, $\nu > -1$. R. G. Cooke [J. London Math. Soc. **12**, 180–185 (1937)] has proved in a rather lengthy way that the areas

$$ \left| \int_{x_{\nu,k-1}}^{x_{\nu,k}} J_\nu(x)dx \right| $$

are increasing with k. The author presents a surprisingly elegant proof for the case $|\nu| > 1/2$ using Sturm's comparison theorem in a proper way. The same method is used in order to prove analogous properties of the Hermite and Legendre polynomials. *G. Szegö* (Stanford, Calif.).

Rainville, E. D. Generating functions for Bessel and related polynomials. Canadian J. Math. **5**, 104–106 (1953).

We employ the usual notation:

$$ \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{u^n}{n!} = {}_pF_q\left[\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}; u\right]. $$

The author defines $\psi_n(x)$ by ${}_{q+2}F_p$ with the parameters $-n, c+n, 1-\beta_i-n$ in the upper row, $1-\alpha_k-n$ in the lower row, $1 \leqq i \leqq q$, $1 \leqq k \leqq p$; $u = (-1)^{p+q+1}x$. The following generating function is proved:

$$ \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \psi_n(x) \frac{t^n}{n!} = R^{-1/2} 2^{c-1} (1+R^{1/2})^{1-c} $$

$$ \times {}_pF_q\left[\begin{matrix} \alpha_1, \cdots, \alpha_p \\ \beta_1, \cdots, \beta_q \end{matrix}; 2t(1+R^{1/2})^{-1}\right]; \quad R = 1-4xt. $$

The $\psi_n(x)$ are generalizations of the "Bessel polynomials" introduced by H. L. Krall and O. Frink [Trans. Amer. Math. Soc. **65**, 100–115 (1949); these Rev. **10**, 453]. Another generalization of the Bessel polynomials and its generating function is also discussed. *G. Szegö* (Stanford, Calif.).

Henrici, Peter. A Neumann series for the product of two Whittaker functions. Proc. Amer. Math. Soc. **4**, 329–334 (1953).

The author expands

$$ M_{k,m}[ir(1+\cos\theta)]M_{k,m}[-ir(1-\cos\theta)](\sin\theta)^{-2m-1} $$

in a series of products of Bessel functions and Gegenbauer polynomials. The expansion correlates solutions of the wave equation in paraboloidal coordinates with those in spherical polar coordinates. *A. Erdélyi* (Pasadena, Calif.).

van Wijngaarden, A. On a certain asymptotic expansion. Quart. Appl. Math. **11**, 244–246 (1953).

This paper is identical with the report reviewed in these Rev. **13**, 842.

Humbert, Pierre. Quelques résultats relatifs à la fonction de Mittag-Leffler. C. R. Acad. Sci. Paris **236**, 1467–1468 (1953).

From the operational image of Mittag-Leffler's function $E_\alpha(x) = \sum_{m=0}^{\infty} x^m/\Gamma(\alpha m+1)$ the author deduces several properties of this function, including an integral equation and relations between $E_\alpha(t^\alpha)$ and $E_{2\alpha}(t^{2\alpha})$. *A. Erdélyi.*

Corrington, M. S. Two non-elementary definite integrals. Math. Tables and Other Aids to Computation **7**, 129–131 (1953).

Hylleraas, Egil. Two-electron angular wave functions. Avh. Norske Vid. Akad. Oslo. I. **1949**, no. 5, 18 pp. (1949).

The author found that certain difficulties in the construction of angular wave functions of a two-electron system are

removed if the angular momenta of the two electrons are quantized separately. He constructs certain angular wave functions as sums of products of spherical surface harmonics.
A. Erdélyi (Pasadena, Calif.).

Colombo, Serge. La fonction de Dirac et son utilisation en physique mathématique. Ann. Télécommun. **8**, 131–144 (1953).

This is an expository article on Dirac's delta function and related topics, giving the operational rules for handling these functions, containing some historical remarks, and justifying the rules by means of singular integrals.
A. Erdélyi (Pasadena, Calif.).

Differential Functions

***Lappo-Danilevsky, J. A. Mémoires sur la théorie des systèmes des équations différentielles linéaires.** Chelsea Publishing Co., New York, N. Y., 1953. xiv+253+208 +204 pp. $10.00.

Photo-offset reprint of Trudy Fiz.-Mat. Inst. Steklov. **5**, x+1–253 (1934); **7**, 1–208 (1935); **8**, 1–204 (1936).

Šul'gin, M. F. On some properties of integrals of ordinary differential equations. Doklady Akad. Nauk SSSR (N.S.) **87**, 701–703 (1952). (Russian)

This paper is a continuation of some earlier ones by the same author [same Doklady (N.S.) **81**, 23–26 (1951); **83**, 373–376 (1952); these Rev. **13**, 594, 996]. Given the system

$$\dot{x}_k = X_k(t; x_1, x_2, \cdots, x_n) \quad (k=1, 2, \cdots, n), \tag{1}$$

introduce superfluous variables y_k and put $H=\sum X_i y_i$. System (1) then may be rewritten in the canonical form

$$\dot{x}_k = \partial H/\partial y_k, \quad \dot{y}_k = -\partial H/\partial x_k. \tag{2}$$

The last n equations, written out in full, are

$$\dot{y}_k + \sum (\partial X_i/\partial x_k) y_i = 0. \tag{3}$$

System (3) has the characteristic property that if

$$f(t; x_1, x_2, \cdots, x_n) = \text{const.}$$

is an integral of (1), then (3) is satisfied by (4) $y_k = \partial f/\partial x_k$. From this property of the system (3) the following theorem follows. If an integral

$$\varphi(t; x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n) = \text{const.}$$

of the extended system (2) is given and the variables y_k are replaced by (4), where $f=$const. is an integral of (1), then $\varphi(t; x_1, x_2, \cdots, x_n; \partial f/\partial x_1, \cdots, \partial f/\partial x_n) =$ const., in general, is a new integral of the system (1). If, in particular, (3) admits an integral which is linear in the y_k, i.e., $\omega = \sum Y_i y_i =$ const., where the Y_i are functions of t; $x_1, x_2, \cdots, x_n$, then also $\omega = \sum Y_i \partial f/\partial x_i =$ const. is an integral of (1). This result leads to a theorem of Buhl [C. R. Acad. Sci. Paris **132**, 313–315 (1901)], which contains the classical theorem of Poisson for canonical systems as a special case. Appell [ibid. **133**, 317–319 (1901)] has shown that Buhl's theorem is a consequence of Poisson's theorem.
E. Leimanis.

Pöschl, Klaus. Zur Frage des Maximalbetrages der Lösungen linearer Differentialgleichungen zweiter Ordnung mit Polynomkoeffizienten. Math. Ann. **125**, 344–349 (1953).

It was shown by Wittich [Math. Ann. **122**, 221–234 (1950); **124**, 277–288 (1952); these Rev. **12**, 500; **14**, 171] that the order of growth of the entire transcendental solutions of the differential equation $w''+c_1(z)w'+c_2(z)w=0$ (the $c_i(z)$ being polynomials) can in general be uniquely obtained from a certain allied quadratic equation. In the present paper, the cases in which Wittich's criterion allowed two possibilities are treated in detail and a refinement of Wittich's criterion which will take care of these cases is given.
Z. Nehari (St. Louis, Mo.).

Wittich, Hans. Eindeutige Lösungen der Differentialgleichungen $w''=P(z, w)$. Math. Ann. **125**, 355–365 (1953).

This paper is concerned with the differential equation

$$w'' = a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z), \quad n \geqq 2, \tag{1}$$

where the $a_i(z)$ are polynomials. It is shown that $n=2, 3$ are the only cases in which (1) can have a solution which is single-valued and non-rational in $|z|<\infty$. Conversely, the author shows that for $n=2, 3$ all solutions of (1) are single-valued and of finite order, provided the coefficients $a_i(z)$ satisfy a certain condition. The value distribution of these solutions is also investigated in detail.
Z. Nehari.

Taam, Choy-Tak. Non-oscillation and comparison theorems of linear differential equations with complex-valued coefficients. Portugaliae Math. **12**, 57–72 (1953).

Earlier investigations by the author [Duke Math. J. **19**, 493–497 (1952); these Rev. **14**, 557] on oscillation of solutions of (1) $W''+Q(x)W=0$ are extended to complex-valued $Q(x)=q_1(x)+iq_2(x)$. He first proves that (1) is disconjugate on an interval I if there exist a real-valued $m(x)$, AC on each closed subinterval of I, and a real constant k, such that $m^1(x)+m^2(x) \leqq -(q_1(x)+kq_2(x))$ p.p. in I. By using this criterion, similarly to Wintner [Amer. J. Math. **73**, 368–380 (1951); these Rev. **13**, 37], a large number of results can be obtained; most of them reduce to previously known results when $Q(x)$ is real; cf. Wintner [loc. cit.] and Borg [ibid. **71**, 67–70 (1949); these Rev. **10**, 456]. It is also shown that (1) is disconjugate on I if $q_1(x)+kq_2(x) \leqq f(x)$ and $y''+f(x)y=0$ is disconjugate on I; this yields extensions of theorems of Hartman and Wintner [ibid. **73**, 885–890 (1951); these Rev. **13**, 652] and Hille [Trans. Amer. Math. Soc. **64**, 234–252 (1948); these Rev. **10**, 456].
G. E. H. Reuter (Manchester).

Taam, Choy-tak. Linear differential equations with small perturbations. Duke Math. J. **20**, 13–25 (1953).

The solutions of $(p(x)y')'+q(x)y=f(x)$ on the interval $(1, \infty)$ are shown to behave, as $x \to \infty$, approximately like those of $y''=0$, provided that $p(x)^{-1}-a$, $q(x)$, and $f(x)$ are small for large x ($a \neq 0$). Typical theorems, extending results of E. Hille [Trans. Amer. Math. Soc. **64**, 234–252 (1948); these Rev. **10**, 376] for the special case when $p(x)=1$, $f(x)=0$, are: (1) If $p(x)^{-1}-a$, $xq(x)$, $xf(x)$ are in $L(1, \infty)$, every solution is $O(x)$, and for each C a solution $y=C+o(1)$ exists; (2) if, in addition, $x^2q(x)$ is in $L(1, \infty)$, every solution has the form $y=Ax+B+o(1)$ for some constants A, B. A number of similar theorems are also proved for the third order equation $(p(x)y'')'+q(x)y'+r(x)y=f(x)$, and for solutions of $(P(z)W')'+Q(z)W=F(z)$ in a sector or semi-infinite strip in the z-plane.
G. E. H. Reuter (Manchester).

Aymerich, Giuseppe. Sulle oscillazioni forzate di due circuiti elettrici non lineari con accoppiamento induttivo e capacitivo. Atti Sem. Mat. Fis. Univ. Modena **5**, 83–89 (1951).

Aymerich, Giuseppe. Oscillazioni forzate periodiche di sistemi non lineari a due gradi di libertà. Atti Sem. Mat. Fis. Univ. Modena **5**, 165–177 (1951).

In the second paper the author proves the existence of periodic solutions for a system

$$\alpha d^2Q/dt^2+dg(Q)/dt+f(Q)=e(t)$$

where Q, g, f, e are two-dimensional vectors, α a symmetric matrix, $e(t+T)=e(t)$, under conditions too complicated to be stated here. The proof is based on a method already used by Graffi [Ann. of Math. (2) **54**, 262–271 (1951); these Rev. **13**, 463] for a similar purpose. A less general problem is discussed in the first paper. *J. L. Massera.*

Jacobsthal, Ernst. Über ein spezielles System simultaner Differentialgleichungen. Norske Vid. Selsk. Forh., Trondheim **25** (1952), 78–81 (1953).

This note relates to the system of differential equations

$$f(x)y_k'=\varphi_k(x)+\sum_{\lambda=1}^{n}a_{k\lambda}y_\lambda \quad (k=1,\cdots,n),$$

where the a's are constants, and the functions f, φ_1, $\cdots$, φ_n possess derivatives of order $n-1$. The typical unknown y_k satisfies a differential equation $L_k(y_k)=\theta_k(x)$, of order n, which can be found by a process of elimination. Here the author shows that the differential operators L_1, $\cdots$, L_n are actually identical. *L. A. MacColl* (New York, N. Y.).

Brock, John E. Variation of coefficients of simultaneous linear equations. Quart. Appl. Math. **11**, 234–240 (1953).

This paper is concerned with the variation of the solution of a system of linear algebraic equations which is induced by a variation of one or more of the coefficients. Matrix formulae are derived, by means of which the variation can be calculated comparatively easily. *L. A. MacColl.*

Höfinger, E. Über die Lösung der gewöhnlichen Laplaceschen Differentialgleichung durch mehrfache Integrale. Monatsh. Math. **57**, 6–18 (1953).

In an earlier paper [Monatsh. Math. **56**, 126–136 (1952); these Rev. **14**, 372] the author constructed a differential equation satisfied by a Mellin convolution. He now similarly constructs a differential equation satisfied by the Laplace convolution of two functions which satisfy ordinary linear differential equations with linear coefficients, and uses his result to discuss the integration of such equations by multiple integrals (like those occurring in integral representations of hypergeometric functions of several variables). *A. Erdélyi* (Pasadena, Calif.).

Duncan, W. J. Solution of ordinary linear differential equations with variable coefficients by impulsive admittances. Quart. J. Mech. Appl. Math. **6**, 122–127 (1953).

The solution $f(t)=\int_0^t Q(\tau)\alpha(t,\tau)d\tau$ of the initial value problem $\sum_{i=0}^{n}P_i(t)f^{(i)}(t)=Q(t)$, $f^{(i)}(0)=0$ $(i=0,1,\cdots,n-1)$, where $\alpha(t,\tau)$ satisfies the differential equation with $Q(t)=0$, for each fixed $\tau<t$ and $\alpha(\tau,\tau)=0$, $\partial\alpha(\tau,\tau)/\partial t=0$, $\cdots$, $\partial^{n-2}\alpha(\tau,\tau)/\partial t^{n-2}=0$, $P_n(\tau)\partial^{n-1}\alpha(\tau,\tau)/\partial t^{n-1}=1$, is pointed out. The corresponding forms of solutions of initial value problems in systems of ordinary linear differential equations are presented. *R. V. Churchill* (Ann Arbor, Mich.).

Otsuki, Tominosuke. On a boundary value problem of systems of paths in plane. Tôhoku Math. J. (2) **4**, 284–293 (1952).

Two theorems are given concerning the possibility of joining two points by a solution of the equation

$$y''=Ay'^3-By'^2+Cy'-D$$

with bounded continuous coefficients. *F. A. Ficken.*

Hayashi, Chihiro. Stability investigation of the nonlinear periodic oscillations. J. Appl. Phys. **24**, 344–348 (1953).

This paper has appeared earlier [Mem. Fac. Eng. Kyoto Univ. **14**, 92–102 (1952); these Rev. **14**, 276].

G. E. H. Reuter (Manchester).

De Castro, A. Soluzioni periodiche di una equazione differenziale del secondo ordine. Boll. Un. Mat. Ital. (3) **8**, 26–29 (1953).

It is shown that if the functions $f(x,\dot{x})$, $g(x)$ satisfy a certain complicated set of rather weak conditions, the equation $\ddot{x}+f(x,\dot{x})\dot{x}+g(x)=0$ possesses at least one periodic solution. *L. A. MacColl* (New York, N. Y.).

Manaresi, Gabriella. Sull'equazione di Liénard generalizzata. Boll. Un. Mat. Ital. (3) **8**, 59–64 (1953).

Assuming that the parameters ϵ, μ are small, and using familiar methods of approximation, the author gives a non-rigorous discussion of the solutions of the equation $\ddot{x}+\omega^2x-\epsilon g'(x)+\mu f(x)\dot{x}=0$. Although the results are of no special mathematical interest, they may well be useful in engineering work. *L. A. MacColl* (New York, N. Y.).

Mizohata, Sigeru, and Yamaguti, Masaya. On the existence of periodic solutions of the non-linear differential equation, $\ddot{x}+a(x)\cdot\dot{x}+\phi(x)=p(t)$. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 109–113 (1952).

The equation $\ddot{x}+a(x)\dot{x}+\phi(x)=p(t)$, with $p(t)$ of period ω and $\int_0^\omega p(t)dt=0$, is shown to have a solution of period ω if $\int_0^x a(u)du\to\pm\infty$ as $x\to\pm\infty$ and $\phi(x)/x\geqq 0$ for $|x|>9$. In addition, if $\phi(x)/x>0$ for $|x|>9$ and $\int_0^x\phi(u)du\to\infty$ as $|x|\to\infty$, every solution satisfies $|x(t)|+|\dot{x}(t)|<B$ (a fixed constant) for sufficiently large t; this last statement is contained in results found independently by the reviewer [Proc. Cambridge Philos. Soc. **47**, 49–54 (1951); these Rev. **12**, 827]. An independent proof is also given that $\ddot{x}+f(\dot{x})+\phi(x)=p(t)$ has a solution of period ω if $f(y)\operatorname{sgn}y\to\infty$ as $|y|\to\infty$ and $\phi(x)\operatorname{sgn}x\to\infty$ as $|x|\to\infty$ [cf. the reviewer, J. London Math. Soc. **27**, 48–58 (1952); these Rev. **13**, 844].

G. E. H. Reuter (Manchester).

Mizohata, Sigeru. On the existence of systems of periodic solutions for several nonlinear circuits. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 115–121 (1952).

The system $\sum_{j=1}^{n}L_{ij}\ddot{x}_j+f_i(x_i)\dot{x}_i+\phi_i(x_i)=p_i(t)$ $(1\leqq i\leqq n)$, with $p_i(t)$ of period ω and $\int_0^\omega p_i(t)dt=0$, is shown to have a solution of period ω under the following conditions: (i) $L_{ij}=L_{ji}$ and $\sum_{i,j}L_{ij}\xi_i\xi_j$ is positive definite; (ii) $\phi_i(x)\operatorname{sgn}x>0$ for $|x|>9$, $\int_0^x\phi_i(u)du\to\infty$ as $|x|\to\infty$; (iii) there exist A and $\epsilon>0$ such that $\sum_{i,j}M_{ij}\phi_i(x_i)[F_j(x_j)-P_j(t)]\geqq\epsilon$ for $x_1^2+\cdots+x_n^2>A^2$, where (M_{ij}) is the inverse matrix of (L_{ij}), $F_j(x_j)=\int_0^{x_j}f_j(u)du$, $P_j(t)=\int_0^t p_j(u)du$. This is shown to imply (a) the theorem of Graffi [Ann. of Math. (2) 54,

262–271 (1951); these Rev. 13, 463], (b) that the system

$$L_1\ddot{x}_1+M\ddot{x}_2+\mu_1(x_1^3-a_1)+k_1x_1=p_1(t),$$
$$M\ddot{x}_1+L_2\ddot{x}_2+\mu_2(x_2^3-a_2)+k_2x_2=p_2(t),$$

has a periodic solution if $M^4/L_1^3L_2^3 \leqq \min(c, c^{-1})$, where $c=\mu_1L_1k_2^3/\mu_2L_2k_1^3$. *G. E. H. Reuter* (Manchester).

Jones, C. W. On reducible non-linear differential equations occurring in mechanics. Proc. Roy. Soc. London. Ser. A. 217, 327–343 (1953).

By means of appropriate changes of variables, the author reduces the consideration of certain solutions of some non-linear differential equations having diverse applications in mechanics to a consideration of corresponding solutions of the equation (*) $dy/dx=y(ax+by+c)/x(a'x+b'y+c')$. The equations discussed are: Emden's equation,

$$y''+2y'/x+y^n=0;$$

the equation $yy''+ay'^2+by=cx$, which the author derives to describe the motion of a piston in a gas expansion chamber; and the equation of Blasius, $f'''+ff''=0$. The pertinent solutions of these equations correspond to separatrices emanating from one or another of the saddle points of (*). The discussion is principally concerned with the topology of the integral curves of (*) and, as regards the equations reduced to (*), aims, in the author's words, "at providing qualitative information which can be used, if necessary, in guiding numerical calculations of the solutions". *C. E. Langenhop* (Ames, Iowa).

Hines, C. O. Reflection of waves from varying media. Quart. Appl. Math. 11, 9–31 (1953).

Consider the wave equation (*) $d^2f/dx^2+k^2n^2f=0$ where $k=2\pi/$(wave length in vacuum) and $n=n(x)$ is the refractive index of the varying medium. Let n^2 be real-valued, finite, and continuous $(-\infty<x<\infty)$ such that $n=1$ for $|x|>x_a>0$ and $n^2\leqq 1$ for $|x|<x_a$. In the interval $|x|<x_a$ the function n^2 is assumed to be "slowly varying" with none, one, or a few regions of negative n^2. By the substitution $f=\exp[ik\int^x r(s)ds]$ equation (*) is transformed into a Ricatti type differential equation of which three solutions are defined, viz., r^+, r^-, and r', by the conditions $r^+=1$ for $x\leqq -x_a$, $r^-=-1$ for $x\leqq -x_a$, and $r'=1$ for $x\geqq x_a$. Setting $r^\pm=\pm r_1+ir_2$, $r'=r_1'+ir_2'$, where r_1, r_2, r_1', r_2' are real-valued, the author discusses in a qualitative manner the character of the curves r_1^2 and $r_2|r_2|$ for refracting media of certain simple types. [The reviewer finds it difficult to follow the author's argument.] Many diagrams illustrate the author's approach to the problem. There is an exhaustive discussion (and construction) of non-reflecting media. The author warns against using other methods (W.K.B., "good-path") especially in the case of two regions of negative n^2. *C. J. Bouwkamp* (Eindhoven).

Burgat, Paul. Résolution de problèmes aux limites au moyen de transformations fonctionnelles. Z. Angew. Math. Physik 4, 146–152 (1953).

A summary of the first three chapters of the author's thesis [Univ. de Neuchâtel, 1950; these Rev. 14, 50].

Morgan, G. W. Some remarks on a class of eigenvalue problems with special boundary conditions. Quart. Appl. Math. 11, 157–165 (1953).

The boundary value problem consisting of the Sturm-Liouville equation $(py')'-qy+\lambda wy=0$ with conditions of the type $y'+K\lambda y=0$ at the endpoints of the interval $x_0\leqq x\leqq x_1$ is reduced heuristically, by using impulse functions, to the classical Sturm-Liouville problem in which the parameter λ does not appear in the boundary conditions. Extensions of the method to self-adjoint differential equations of higher even orders are pointed out, and formulas for the expansion of an arbitrary function in series of the characteristic functions are given. Applications to engineering problems are cited. Reviewer's note: rigorous treatments of such expansion problems are given by J. Tamarkin [Math. Z. 27, 1–54 (1927)], W. T. Reid [Amer. J. Math. 54, 769–790 (1932)], and by the reviewer [Bull. Amer. Math. Soc. 48, 143–149 (1942); these Rev. 3, 235]. *R. V. Churchill* (Ann Arbor, Mich.).

Greco, Donato. Sulla convergenza degli sviluppi in serie di autosoluzioni associati ad un problema ai limiti relativo ad un'equazione differenziale ordinaria del secondo ordine. Rend. Accad. Sci. Fis. Mat. Napoli (4) 17 (1950), 171–189 (1951).

The author perfects his earlier treatment [Giorn. Mat. Battaglini (4) 2(78), 216–237 (1949); 3(79), 86–120 (1950); these Rev. 11, 437; 12, 103] by obtaining asymptotic values for the eigenvalues and eigenfunctions, thereby enabling him to treat the convergence of the eigenfunction expansion of functions. *N. Levinson* (Cambridge, Mass.).

Oleĭnik, O. A., and Žižina, A. I. On a boundary problem for the equation $\epsilon y''=F(x, y, y')$ for small ϵ. Mat. Sbornik N.S. 31(73), 709–717 (1952). (Russian)

It is assumed that the equation $\epsilon y''+A(x, y)=f(x, y, y')$ has a solution $y_\epsilon(x)$ over $[x_0, x_1]$ with $y_\epsilon(x_0)=y_0$ and $y_\epsilon(x_1)=y_1$ for any small $\epsilon>0$. It is further assumed that there is a constant L such that $|y_\epsilon(x)|<L$. The functions A and f are assumed to be of class C' in (x, y, y') for $x_0\leqq x\leqq x_1$, $|y|<L$ and $-\infty<y'<\infty$ and in this region $A-\partial f/\partial y'\geqq K>0$ for some constant K. For any $\alpha>0$, over the interval $[x_0+\alpha, x_1]$ $y_\epsilon(x)$ tends uniformly, as $\epsilon\to 0$, to the solution $y(x)$ of $Ay'=f$ which at x_1 assumes the value y_1. Moreover, $y_\epsilon'(x)$ tends uniformly to $y'(x)$ over the same interval as $\epsilon\to 0$. *N. Levinson* (Cambridge, Mass.).

Bargmann, V. On the number of bound states in a central field of force. Proc. Nat. Acad. Sci. U. S. A. 38, 961–966 (1952).

The Schrödinger equation $\phi''-l(l+1)r^{-2}\phi+E\phi=V(r)\phi$ is considered on $0\leqq r<\infty$ with integer $l\geqq 0$. The integral $I=\int_0^\infty r|V(r)|dr$ is assumed to be finite. The number of bound states (i.e., the number of values of the parameter E for which the boundary-value problem $\phi(0)=0$ has solutions of class $L^2(0, \infty)$) is denoted by n_l. It is proved that $n_l<I/(2l+1)$, which generalizes the result of Jost and Pais who showed for the case $l=0$ that no bound states occur when $I<1$. *N. Levinson* (Cambridge, Mass.).

Laurikainen, K. V. Asymptotic eigensolutions of the radial deuteron equation. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 130, 10 pp. (1952).

The eigenvalues b and the eigenfunctions Φ for the differential equation

$$\frac{d^2\Phi}{dx^2}-\left(k^2+\frac{l(l+1)}{x^2}-b\frac{e^{-x}}{x}\right)\Phi=0$$

and the boundary conditions $\Phi(0)=\Phi(\infty)=0$ are calculated by an asymptotic method for large k. When $l=0$ or 1, the numerical results obtained are accurate to three decimals

for k as small as three. For the asymptotic solution of the problem the eigenfunctions are written in the form $\Phi(x)=e^{-kx}x^{l+1}\sum_{j=0}^{\infty}\omega_j(k,b)x^j$, and in the resulting recursion formula for the $\omega_j(k,b)$ the formal series $b=\sum_{r=-1}^{\infty}\beta_r\tau^r$, $\omega_j=\sum_{r=0}^{\infty}\omega_{jr}\tau^r$, with $\tau=(2k+\sigma)^{-1}$ are inserted. The β_r and ω_{jr} can then be calculated successively. The constant σ is chosen so as to have $\beta_0=0$. No proof is given for the asymptotic character of the resulting series. *W. Wasow.*

Dorodnicyn, A. A. Asymptotic laws of distribution of the characteristic values for certain special forms of differential equations of the second order. Uspehi Matem. Nauk (N.S.) 7, no. 6(52), 3–96 (1952). (Russian)

The author discusses the boundary-value problem

$$y''+[\lambda^2r(x)+q(x)]y=0, \quad a\leqq x\leqq b,$$
$$y(a)\cos\alpha-y'(a)\sin\alpha=0,$$
$$y(b)\cos\beta+y'(b)\sin\beta=0, \quad \alpha,\beta \text{ const.},$$

in certain cases when the classical Sturm-Liouville theory does not apply. There are four such cases:

(i) $r(x)=(x-a)r_1(x)$,

(ii) $r(x)=(x-a)(b-x)r_1(x)$,

(iii) $r(x)=(x-c)r_1(x), \quad a<c<b$,

(iv) $r(x)=(x-a)^\rho r_1(x), \quad \rho>-2$.

In all four cases it is assumed that $r_1(x)$ and $q(x)$ are bounded continuous functions on the closed interval (a, b), and $r_1(x)$ is bounded away from zero there. Using Langer's method for finding asymptotic forms of the solutions of the differential equation for large λ, the author finds asymptotic forms of the characteristic values and characteristic functions of the boundary-value problems. Using these, he discusses the convergence of the bilinear expansions of the Green's functions of his problems. The theory is applied to several differential equations among which Mathieu's equation may be mentioned as being of considerable interest.

The author also discusses in a similar manner a singular Sturm-Liouville problem in which the differential equation is of the form

$$xy''+p(x)y'+[\lambda^2r(x)+q(x)]y=0$$

p, q, r are continuous on the closed interval $(0, l)$, $r(x)$ is bounded away from zero there, no boundary condition may be prescribed at $x=0$, and the boundary condition at $x=l$ is $y(l)\cos\lambda+y'(l)\sin\lambda=0$.

In the appendix various properties of the comparison functions (Bessel functions) are presented; these are not new. *A. Erdélyi* (Pasadena, Calif.).

Sarginson, K. The expansion of a four-dimensional plane wave in a series of four-dimensional pseudo harmonics. J. London Math. Soc. 28, 21–31 (1953).

The Klein-Gordon equation $\Delta u-u_{tt}+P^2u=0$ can be solved by separation of variables in the following coordinate system: $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$, $r=R\cosh\alpha$, $t=R\sinh\alpha$. The author derives the expansion of a plane wave $\exp i(p_1x+p_2y+p_3z-Et)$ in terms of these solutions. The results are complicated because the range for α is made up of two separate lines: one is the real axis from $-\infty$ to $+\infty$ and this determines points inside the characteristic cone; the other is a parallel to the real axis, the line from $\frac{1}{2}\pi i-\infty$ to $\frac{1}{2}\pi i+\infty$, and this determines points outside the characteristic cone. The required expansions are obtained by the use of the methods of Titchmarsh [Eigenfunction expansions . . . , Oxford, 1946; these Rev. 8, 458]. In the course of the work, certain integrals involving associated Legendre functions are evaluated. The following is typical:

$$\int_0^\infty \frac{Q_l^k(\coth y)}{\sinh y}\,\frac{J_{l+1/2}(R\sinh y\sinh B)}{(R\sinh y\sinh B)^{1/2}} \times\exp(iR\cosh y\cosh B)\sinh^2 y\,dy = (\pi/2)^{1/2}\exp\{(k+l+2)\pi i/2\}\frac{Q_l^k(\coth B)}{\sinh B}\,\frac{H_k^{(1)}(R)}{R}.$$

B. Friedman (New York, N. Y.).

***Reeb, G. Sur la stabilité et l'unicité des solutions périodiques de l'équation différentielle**

$$X(x,y)dx+Y(x,y)dy=0.$$

Colloque de Topologie de Strasbourg, 1951, no. X, 8 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1952.

The author shows how the characteristic exponent of a closed solution of $Xdx+Ydy=0$ can be found by integration of an appropriate differential form. This is applied to deduce several stability criteria, in particular, for Liénard's equation: $x''+f(x)x'+g(x)=0$. *W. Kaplan.*

***Bergman, Stefan, and Schiffer, M. Kernel functions and elliptic differential equations in mathematical physics.** Academic Press Inc., New York, N. Y., 1953. xiii+432 pp. $8.00.

In this book the authors collect their researches of the last few years on elliptic partial differential equations. The first part of the book is devoted largely to background material on heat conduction, fluid dynamics, electrostatics, and elasticity, together with the more formal applications of variational formulas and the kernel function. The second part lays more stress on rigor, and treats fundamental solutions, reduction of boundary value problems to integral equations, orthonormal systems and kernel functions, eigenvalue problems associated with the kernels, variational theory of domain functions, comparison domains, basic existence theorems, and dependence of solutions on the boundary data or on the coefficients of the differential equation. The presentation is in an easy flowing style, and the material should prove to be a most useful guide to those interested in the more advanced theory of linear elliptic partial differential equations. Ample references to further literature are provided, although unfortunately the contribution of D. Riabouchinsky [C. R. Acad. Sci. Paris 185, 840–841 (1927)] on free boundary flows is not mentioned. *P. R. Garabedian* (Stanford, Calif.).

Tautz, Georg L. Zum Umkehrungsproblem bei elliptischen Differentialgleichungen. I, II. Arch. Math. 3, 232–238, 239–250 (1952).

Under certain general conditions the author establishes the existence of a unique linear homogeneous partial differential equation of second order having preassigned solutions of the first boundary value problem. In a bounded region T of n space, $n\geqq 3$, consider a system of regions ω among which are the regions with piecewise smooth boundaries. To every ω and every continuous function f defined on the boundary of ω is assigned a unique twice continuously differentiable function $R_{f,\omega}(P)$ that assumes the values f on the boundary of ω. These functions, which are called regular, satisfy the following conditions: (1) if $\omega^*\subset\omega$ and $R_{f,\omega}(P)=u(P)$, then $R_{u,\omega^*}=u(P)$; (2) $R_{f,\omega}$ is linear in f; (3) the maximum principle: $R_{f,\omega}(P)\leqq\text{Max}(\text{Max } f, 0)$, equality holding only

if f is constant; (4) for every point $Q \in T$ there is a function $G(P, Q)$ vanishing on the boundary of T and regular in T except at Q where it becomes infinite as $\overline{PQ}^{-n}$; (5) the integral equation $v(P)=\int_T G(P, Q)\alpha(Q)dQ$ has a unique continuous solution $\alpha(Q)$ for every $v(P)$ twice continuously differentiable in T, continuous in $\bar{T}$, and vanishing on the boundary of T. Under these and certain regularity conditions omitted here the author proves in Part I the existence of a unique linear homogeneous elliptic partial differential equation of the second order having the regular functions as solutions and $G(P, Q)$ as its Green's function. In Part II the author shows that with slight sharpening of the conditions on $G(P, Q)$ the same theorem is true without condition (5); however, there is now a certain nowhere dense closed set in T on which the regular functions might possibly not satisfy the differential equation of the theorem. *D. Gilbarg* (Bloomington, Ind.).

Tautz, G. L. Bemerkungen zu meiner Arbeit: Zum Umkehrungsproblem bei elliptischen Differentialgleichungen, I, II. Arch. Math. 3, 361–365 (1952).

The author shows that in the theorem of the preceding review the maximum principle (condition (3)) is superfluous. He also gives a proof of the following theorem, which he attributes to Delassus but cannot find in the literature: Any linear differential equation of the second order with continuous coefficients which possesses a Green's function must be of elliptic type. The Green's function of this theorem is required to possess certain regularity properties stated by the author. *D. Gilbarg* (Bloomington, Ind.).

Kamenomostskaya, S. L. On equations of elliptic and parabolic type with a small parameter in the highest derivatives. Mat. Sbornik N.S. 31(73), 703–708 (1952). (Russian)

The author points out that the reviewer in his treatment of the Dirichlet problem for $\epsilon\Delta u+Au_x+Bu_y+Cu=D$, ϵ small, in a multiply connected domain R [Ann. of Math. (2) 51, 428–445 (1950); these Rev. 11, 439] does not include the case where the boundaries of R consist exclusively of closed integral curves of the equation $dx/A=dy/B$. The author treats this case with $C<0$ and under the assumption that the degenerate equation $Au_x+Bu_y+Cu=D$ has a smooth single-valued solution in all of R. In this case boundary-layer terms may occur at all boundaries. If the solution of the Dirichlet problem is $u_\epsilon(x, y)$ and that of the degenerate equation is $u(x, y)$, then $u_\epsilon=u+z+O(\epsilon)$ where z is made up of boundary-layer terms. Thus in the case $D=0$, for example, then $u\equiv 0$ and the result is that the solution of the Dirichlet problem tends to zero inside of R as $\epsilon\to+0$. The proof here is simpler than in the case cited above since, where R has this special type of boundary, only the maximum principal need be used. The equation

$$\epsilon u_{xx}+Au_t+Bu_x+Cu=D$$

is also treated. *N. Levinson* (Cambridge, Mass.).

Kahan, Théo. Sur les méthodes variationnelles dans la théorie des collisions et de la diffraction. C. R. Acad. Sci. Paris 236, 1000–1003 (1953).

The author points out that the variational principle of Schwinger is a special case of one given by Courant and Hilbert [Methoden der mathematischen Physik, vol. 1, 2nd ed., Springer, Berlin, 1931, p. 176]. *B. Friedman.*

Kupradze, V. D. Fundamental problems in the mathematical theory of diffraction (steady state processes). Translated by C. D. Benster. U. S. Department of Commerce, National Bureau of Standards, Los Angeles, Calif., NBS Rep. 2008, viii+160 pp. (1952).

Translated from the author's Osnovnye zadači matematičeskoĭ teorii diffrakcii (ustanovivšiesya processy) [Glav. Redakciya Obščetehn. Lit., Leningrad-Moscow, 1935].

Keller, Joseph, B. The scope of the image method. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. BR-2, i+10 pp. (1953).

The image method to which reference is made in the title is a method for constructing a Green's function for a part D of space bounded by planes in terms of the corresponding Green's function for the full space, the boundary condition being the vanishing of the normal derivative. The first requirement is evidently that the set of images of the singularity of the Green's function, the images of the images, etc., can have only one point, viz., the singularity itself, in the region D considered. The reflections in the planes bounding D form a group, which must be discrete and have D as a fundamental domain. The fundamental domains of discrete groups of reflections are known; but only those for which the angle between any two bounding planes is of the form π/p (p an integer) have the desired property. The second problem is to determine the linear second order partial differential equations to which the method is applicable. The author appears to assume in this part of the work that the differential equation has constant coefficients. Lastly, the investigation is extended to the boundary conditions $G=0$ or $\partial G/\partial n+hG=0$. *E. T. Copson.*

Moshinsky, Marcos. On a class of transformations of interest in nuclear scattering. Revista Mexicana Fisica 1, 114–126 (1952). (Spanish. English summary)

The physical problem leads to the partial differential equation $-\frac{1}{2}\phi_{rr}+V(r)\phi=i\phi_t$, where $\phi=\phi(r, t)$, and $\phi(r, 0)=f(r)$ is given for $r>0$. The potential $V(r)$ is assumed to be continuously differentiable, with sectionally continuous second derivatives, and to vanish when $r>a$. Under the assumption $f(r)=0$ when $r\leqq a$, the author represents $\phi(r, t)$ for $r>a$ in terms of the Green's function of the problem and points out that this Green's function may be used as the kernel of a functional transformation. *A. Erdélyi.*

***Laasonen, P. Das Wärmeleitungsproblem einer linearen Mannigfaltigkeit.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 148–152. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

The author considers the initial value problem for the heat equation $u_t=u_{xx}$ in a one-dimensional complex N consisting of a finite number of bounded arcs with element of length dx. At each vertex the sum of the x-derivatives of u in the directions away from the vertex shall vanish. A solution of the problem by Laplace transformation is sketched. It is based on construction of the Green's function for the resulting ordinary differential equation on N from that belonging to a simply connected covering graph of N. *F. John* (New York, N. Y.).

Kampé de Fériet, Joseph. Un théorème d'unicité pour les intégrales de l'équation de la chaleur appartenant à la classe L. C. R. Acad. Sci. Paris **236**, 1527–1529 (1953).

A uniqueness theorem for the equation $u_{xx}=u_t$ for $t>0$, $-\infty<x<\infty$, and with initial conditions on $t=0$ is proved. Uniqueness has been proved under much weaker conditions by various authors [inter alia, Cooper, J. London Math. Soc. **25**, 173–180 (1950); these Rev. **12**, 104].

J. L. B. Cooper (Cardiff).

***Gårding, Lars. Le problème de Goursat pour l'équation des ondes.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 255–258. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

The author gives a solution of the problem of finding a function $f(x_1, x_2, x_3, x_4)$, which satisfies the equation

$$f_{x_1x_1}-f_{x_2x_2}-f_{x_3x_3}-f_{x_4x_4}=g(x_1, x_2, x_3, x_4)$$

inside a time-like cone S, and vanishes on S. [Reference is made to an earlier solution by Soboleff, Mat. Sbornik N.S. **11**(53), 155–203 (1942); these Rev. **5**, 8.] The problem is reduced to an integro-differential equation for the normal derivative of f on S. This latter equation is solved by iteration for a special type of function g; the solution is then extended to general g of class C^3 and of suitable behavior at the vertex of the cone by establishing a priori estimates between g and the normal derivative of f. *F. John.*

Gårding, Lars. The asymptotic distribution of the eigenvalues and eigenfunctions of a general vibration problem. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 109–118 (1952).

This paper appeared earlier in Kungl. Fysiografiska Sällskapets: Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **21**, no. 11 (1951); these Rev. **14**, 653.

Vallander, S. V. On nonlinear hyperbolic partial differential equations of the second order. Doklady Akad. Nauk SSSR (N.S.) **89**, 201–204 (1953). (Russian)

As a generalization of a previous paper [same Doklady (N.S.) **83**, 637–639 (1952); these Rev. **13**, 847] the following theorem is proved: necessary and sufficient conditions that a hyperbolic equation

$$(1)\qquad F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x\partial y}, \frac{\partial^2 u}{\partial y^2}\right)=0$$

have a solution of the form

$$(2)\qquad u=\Omega\{x, y, \Phi_1[\alpha(x, y)], \Phi_2[\beta(x, y)]\},$$

where Ω, α, β are certain fixed functions and Φ_1 and Φ_2 are arbitrary, are: (I) the characteristics do not depend explicitly on the solution function u; (II) after transforming to the normal form by the characteristic equations $\xi=\alpha(x, y)$, $\eta=\beta(x, y)$ and solving for the mixed derivative, equation (1) must have the form

$$(3)\qquad \frac{\partial^2 u}{\partial\xi\partial\eta}+A\frac{\partial u}{\partial\xi}+B\frac{\partial u}{\partial\eta}+C\frac{\partial u}{\partial\xi}\frac{\partial u}{\partial\eta}+D=0,$$

where A, B, C, and D are certain functions of (ξ, η, u); (III) the coefficients A, B, C, and D must satisfy the relations

$$(4)\qquad \frac{\partial A}{\partial\xi}+AB=\frac{\partial D}{\partial u}+CD,\quad \frac{\partial A}{\partial u}=\frac{\partial C}{\partial\eta},$$
$$\frac{\partial B}{\partial\eta}+AB=\frac{\partial D}{\partial u}+CD,\quad \frac{\partial B}{\partial u}=\frac{\partial C}{\partial\xi}.$$

Further, if conditions (I), (II), and (III) are satisfied, then the general solution of equation (3) is found implicitly in a relation $N(\xi, \eta, u)=\Phi_1(\xi)+\Phi_2(\eta)$, where Φ_1 and Φ_2 are arbitrary and N is a function found from the relation $\partial N/\partial u=\exp\int(Ad\eta+Bd\xi+Cdu)$. *C. G. Maple.*

Integral Equations

Satō, Tokui. Sur la limitation des solutions d'un système d'équations intégrales de Volterra. Tôhoku Math. J. (2) **4**, 272–274 (1952).

The basic lemma of this note states that if $u(x)$ satisfies the integral equation $u(x)=f(x)+\int_a^x K(x, t, u(t))dt$, where $f'(x)$ exists on I: $a\leqq x<\infty$, $K(x, t, u)$ is continuous for x on I and (t, u) in a domain ϑ, $K_x'(x, t, u)$ continuous, $\geqq 0$ and monotone in u, if further $\omega(x)$ satisfies the inequalities

$$\omega'(x)>f'(x)+K(x, x, \omega(x))+\int_a^x K_x'(x, t, \omega(t))dt$$

and $\omega(a)>f(a)$, then $\omega(x)>f(x)$ for all x. This is used to prove that if $u_j(x)$ is a system of solutions of the equations $u_j(x)=\sum_k^n\int_a^x a_{jk}(x, t)u_k(t)dt+b_j(t)$, a_{jk} continuous on $a\leqq t\leqq x<\infty$, b_j continuous and bounded on I, then

$$\sum_j|u_j(x)|\leqq B\exp\int_a^x A(x, t)dt,$$

where $\max_{jk}|a_{jk}(x, t)|=A(x, t)$ and $\sum_j|b_j(x)|\leqq B$. [See Butlewski, Studia Math. **10**, 40–47 (1948); these Rev. **9**, 435.] *T. H. Hildebrandt* (Ann Arbor, Mich.).

Ullman, J. L. On a theorem of Frobenius. Michigan Math. J. **1** (1952), 189–193 (1953).

Let $A=[a_{ij}]$ be an $n\times n$ matrix such that $a_{ij}\geqq 0$ for all (i, j). The author calls a sequence of r elements of A of the form $a_{i_1i_2}, a_{i_2i_3}, \cdots, a_{i_ri_1}$ a cycle, and shows that A has a positive eigenvalue if and only if it has a cycle consisting entirely of non-zero elements. By a similar argument he shows that if $K(s, t)$ is a continuous non-negative kernel defined in $a\leqq s\leqq b$, $a\leqq t\leqq b$, then it has a positive characteristic value if and only if there is a cycle of points (t_1, t_2), (t_2, t_3), $\cdots$, (t_r, t_1) in its square of definition at all of which $K(s, t)$ is non-zero. *F. Smithies* (Cambridge, England).

Parodi, Maurice. Sur certaines équations intégrales fonctionnelles. C. R. Acad. Sci. Paris **236**, 1729–1731 (1953).

In this paper the author develops a formal solution of the equation

$$f(at)+\lambda\int_0^\infty k(x, t)f(x)dx=g(t)$$

[$k(x, t)$, $g(t)$, λ, a given] by means of operational calculus, assuming that the Laplace transform of $k(x, t)$ (with respect to t) is of the form $\rho(s)e^{-s\phi(s)}$. He illustrates his result by taking the Hankel kernel as $k(x, t)$. *A. Erdélyi.*

Maksimov, I. M. On sum equations. Doklady Akad. Nauk SSSR (N.S.) **89**, 401–403 (1953). (Russian)

The equation (1) $\varphi(x)=\lambda\sum_{s=1}^n K(x, s)\varphi(s)+f(x)$ in the function $\varphi(x)$ is called by the author a sum equation, and its importance derives from the result that every system of linear algebraic equations in infinitely many unknowns can be transformed into a sum equation of form (1) ($\lambda=1$) without altering the set of solutions. That (1) is analogous

in form to the Fredholm integral equation of the second kind is apparent; the results of the present work show that the analogy is quite deep. Equation (1) is called regular when the following conditions hold: $\sum_{s=1}^{\infty}|K(x,s)|^{\delta}\leqq K$, $\sum_{s=1}^{\infty}|f(x)|^{\delta}\leqq f$ ($\delta=1$ or 2), and solutions of (1) are sought which satisfy $\sum_{s=1}^{\infty}|\varphi(x)|^{\delta}\leqq\varphi$. Here K, f, φ are constants. For a regular equation (1), the Fredholm apparatus (entire functions $D(\lambda)$, $D(x,y,\lambda)$, etc.) is set up, and it yields the exact analogues of the fundamental Fredholm theorems. All results are stated without proof. *I. M. Sheffer.*

Gegeliya, T. G. On some singular integral equations of particular form. Soobščeniya Akad. Nauk Gruzin. SSR 13, 581–586 (1952). (Russian)

Let L be a simple, closed, rectifiable Jordan curve and let $\phi(t)$ be continuous, possibly complex-valued, on L; $\phi(t)\in B$ if the limit

$$\lim_{\delta\to 0}\frac{1}{2\pi}\int_{L-\delta(t)}\left|\frac{\phi(\tau)-\phi(t)}{\tau-t}\right||d\tau|$$

exists, uniformly with respect to t on L; here $\delta(t)$ is an arc of L of length δ and of center t. If $\phi\in B$ and

$$\Phi(t)=\frac{1}{2\pi i}\int_L\frac{\phi(\tau)-\phi(t)}{t-\tau}d\tau\in B,$$

ϕ is said to be in B_1. A curve L belongs to the class A if for some $k>0$ one has $\rho(t',t'')$ [=distance between t', t''] $\geqq kS(t',t'')$ for all points t', t'' on L, $S(t',t'')$ being the length of the least arc of L bounded by t', t''. Some of the results are as follows. If $L\in A$ and $\phi\in H$ (Hölder), then $\phi\in B_1$. If $L\in A$ and $\phi\in H(\mu)$, then $\Phi(t)\in H(\mu)$ (for $\mu<1$), $\Phi(t)\in H(1-\epsilon)$ (for $\mu=1$). Let now $L=\sum_0^n L_k$, where the L_k are disjoint, closed Jordan curves and L_0 contains the L_i ($i>0$). Designate by S^+ the bounded, connected domain, limited by L, and let S^- be the complement of S^++L. If $L_0,\cdots,L_n$ are in A and $\phi\in H$, then Φ is continuously extendable on L from S^+ (from S^-), while Φ^+, Φ^- belong to H. A study along familiar lines is made of the integral equation in the sense of principal values:

$$(1)\qquad M\phi\equiv a(t)\phi(t)+\frac{b(t)}{2\pi i}\int_L\frac{\phi(\tau)-\phi(t)}{\tau-t}d\tau=f(t);$$

here $L=\sum_0^n L_j$, $L_j\in A$; f, a, $b\in H$; $a(t)[a(t)-b(t)]\neq 0$; ϕ is the unknown in H; (1) is equivalent to a suitable functional boundary problem. Finally the author considers the equation

$$(2)\qquad M\phi+m\phi\equiv M\phi+\frac{1}{2\pi i}\int_L m(t,\tau)\phi(\tau)d\tau=f(t),$$

where $m\phi\in H$ whenever $\phi\in H$. Necessary and sufficient conditions are established under which (2) is equivalent to a regular Fredholm integral equation.

W. J. Trjitzinsky (Urbana, Ill.).

****Kourganoff, V. Basic methods in transfer problems. Radiative equilibrium and neutron diffusion. With the collaboration of I. W. Busbridge.** Oxford, at the Clarendon Press, 1952. xv+281 pp. $7.00.

The main body of this book (Chapters 3–6) is devoted to a detailed account of the various methods which have been developed in the past for the solution of the equation of transfer

$$(*)\qquad \mu\frac{dI(\tau,\mu)}{d\tau}=I(\tau,\mu)-J(\tau);\quad J(\tau)=\frac{1}{2}\int_{-1}^{+1}I(\tau,\mu')d\mu'$$

together with the boundary conditions $I(0,-\mu)=0$, $0<\mu\leqq 1$ and $J(\tau)\to\epsilon^{(1-\epsilon)\tau}$ ($\epsilon>0$) for $\tau\to\infty$. The problem is equivalent to solving the Schwarzschild-Milne integral equation

$$J(\tau)=\frac{1}{2}\int_0^{\infty}J(t)E_1(|t-\tau|)dt,$$

where $E_1(x)$ denotes the first exponential integral. The methods which are described are: (i) the early "intuitive" methods of Milne and Eddington; (ii) the method based on the expansion of $I(\tau,\mu)$ in Legendre polynomials in μ; (iii) the method of discrete ordinates which replaces (*) by a system of linear differential equations; (iv) iterative methods based on the Schwarzschild-Milne integral equation; (v) variational methods; and (vi) methods derived from principles of invariance. While the methods are instructive and the authors' manner of treating them on a "competitive" basis highly interesting, it is in fact not true that all the methods described are of equal (or even, comparable) value in attacking transfer problems of even slightly greater generality than that to which the book is devoted. If one goes to problems of radiative transfer which involve scattering in accordance with a phase function or a phase matrix or even the isotropic problem in atmospheres of finite optical thicknesses, so far, only methods (iii) and (vi) have led to any definite results: Exact solutions of any problem besides that of isotropic scattering in semi-infinite atmospheres have not been obtained by any of the other methods described in the book.

The last chapter is devoted to the problem of radiative equilibrium of a stellar atmosphere in local thermodynamic equilibrium and in which the absorption coefficient is dependent on frequency. After reading this chapter one is left with the impression that the only reliable method of solving this problem is a numerical one based on iteration: but convergence of any such iterative process is likely to be very slow and, moreover, from the extent of the numerical agreement between the iterates one cannot infer that the solution has been determined to a corresponding accuracy.

S. Chandrasekhar (Williams Bay, Wis.).

Nishimura, Jun, and Kamata, Koichi. On the theory of cascade showers. I. Progress Theoret. Physics 7, 185–192 (1952).

The usual method of solving the diffusion equations of the cosmic ray cascade showers (see eqs. (1) and (2) of the following review) is to apply a Mellin transformation (with respect to E) to the equations, and obtain the solution as the inverse Mellin transform as a power series in (β/E). [This is the method of Bhabha and Chakrabarty, Physical Rev. (2) 74, 1352–1363 (1948).] The radius of convergence of this latter series is such that it cannot be usefully adapted to the computation of the required functions for all values of the arguments which are of interest. In this paper the author shows how by a process of analytic continuation applied to the function represented by the series within the radius of convergence, the problem of the numerical evaluation of the cascade functions can be made more practicable. The author's method of analytic continuation enables one to recover also the solution in the form obtained by Scott [ibid. 80, 611–615 (1950); these Rev. 12, 832].

S. Chandrasekhar (Williams Bay, Wis.).

Monticelli, F. Sui metodi di soluzione delle equazioni della cascata elettrofotonica. Nuovo Cimento (9) **9**, 477–486 (1952).

The diffusion equations which one has to solve in the theory of cosmic ray cascade showers are

$$(1)\quad \frac{\partial \pi(E,t)}{\partial t}-\beta\frac{\partial \pi(E,t)}{\partial E}=\int_E^\infty \pi(E',t)R(E',E'-E)\frac{E'-E}{E'^2}dE'-\pi(E,t)\int_0^E R(E,E')\frac{E'}{E^2}dE'+2\int_E^\infty \gamma(E',t)R(E,E')\frac{dE'}{E'}$$

and

$$(2)\quad \frac{\partial \gamma(E,t)}{\partial t}=\int_E^\infty \pi(E',t)R(E',E)\frac{E}{E'^2}dE'-D\gamma(E,t),$$

where

$$R(E',E)=1-\left(\alpha+\frac{4}{3}\right)\frac{E'}{E}\left(1-\frac{E'}{E}\right),\quad D=\frac{7}{9}-\frac{\alpha}{6}$$

and α is a constant. By applying a Laplace transformation with respect to t (instead of, as usual, a Mellin transformation with respect to E) to the foregoing equations the author derives for the Laplace transform $L_\pi(E,\lambda)$ of $\pi(E,t)$ the following Volterra's integral equation of the second kind:

$$(3)\quad \beta L_\pi(E,\lambda)=\int_E^\infty \pi(E',0)dE'-\int_E^\infty L_\pi(E',\lambda)K(E',E)dE'$$

where the nucleus $K(E',E)$ is given by

$$K(E',E)=\lambda+\int_0^{E'} R(E',x)\frac{x}{E'^2}dx-\int_E^{E'} R(E',E'-x)\frac{E'-x}{E'^2}dx-\frac{2}{\lambda+D}\int_E^{E'}\bar{R}(E',x)\frac{dx}{E'}$$

and

$$\bar{R}(E',E)=\int_E^{E'} R(E',\eta)R(E,\eta)d\eta/E'.$$

The author shows how the standard methods known in the theory of integral equations when applied to (3) leads to the same series solution (in power of β/E) which has been obtained earlier by Bhabha and Chakrabarty [Physical Rev. (2) **74**, 1352–1363 (1948)]. *S. Chandrasekhar.*

Functional Analysis, Ergodic Theory

***Bourbaki, N. Eléments de mathématique. XV. Première partie: Les structures fondamentales de l'analyse. Livre V: Espaces vectoriels topologiques. Chapitre I: Espaces vectoriels topologiques sur un corps valué. Chapitre II: Ensembles convexes et espaces localement convexes.** Actualités Sci. Ind., no. 1189. Hermann & Cie, Paris, 1953. ii+124+iv pp.

This book is the first two chapters of an exposition of the theory of linear topological spaces. The first chapter treats linear topological spaces over a field with an absolute value. Assuming only this rather general hypothesis, the authors proceed to localize, in the usual way, the notions of topology, continuity and equicontinuity (of multilinear functions), and uniform structure. Product spaces, subspaces and complementary subspaces are discussed briefly and concisely. If the underlying field is complete and non-discrete, it is shown that a finite-dimensional Hausdorff space necessarily has the product topology, and that the space is locally compact if and only if it is finite-dimensional and the field is locally compact. The open mapping theorem and the closed graph theorem are proved for metrizable complete spaces over the same sort of field.

The second chapter is devoted to convex sets and to the elementary theory of locally convex spaces. The scalar field is supposed to be the reals, except in the last section where an extension to complex spaces is made. After geometric preliminaries, and exhibition of the correspondence between cones and partial orders, the authors consider locally convex spaces. A relatively new device for constructing new topologies from old is described: if, for each $a \in A$, f_a is a linear map of a linear topological space E_a into a space E, one may consider the finest (=largest=strongest) locally convex topology for E which makes each f_a continuous. (Observe that without the requirement of local convexity, this stipulation does not usually give a linear topological space.) One or two simple properties of this construction are proved—apparently as background for the eventual study of LF spaces and the projective-inductive limit (=inverse-direct limit) duality. Separation theorems are then proved, based on the lemma that a linear variety which is disjoint from a convex open set and maximal with respect to this property is a closed hyperplane. (The classical Hahn-Banach theorem shows up a little later as an exercise (16, p. 105).) In particular, a theorem on the extension of positive linear functionals is proved. The authors then consider compact subsets (the convex envelope is compact if the space is complete Hausdorff) and prove the classic Krein-Milman theorem on the existence of extreme points. Finally, the equivalence of a locally convex topology to a family of semi-norms (=pseudo-norms) is established, and a number of results are extended to complex spaces. In an appendix, the Markoff-Kakutani fixed point theorem for a commuting family of affine maps on a convex compact subset of a linear topological space is proved, and as an application, the existence of Haar measure on a compact Abelian group is derived.

Summing up, the book is a careful, extremely well-organized account of the foundations of the theory, covering essentially the elementary topology and geometry prerequisite to the duality theory which forms the core of the subject. The exposition is very well documented with examples and problems. Some of the problems (e.g., 9, p. 68) are more profound and, in my opinion, more useful than results which occur in the text; it is unfortunate that such results are interred so obscurely. Other criticism (perhaps a matter of taste) is possible. For example, in the treatment of the open mapping theorem, results could be improved by use of the sharper lemma (which is proved but not stated): Let f be a function (or even a relation) with domain a metric space X and range a subset of a metric space Y such that: a) the graph of f is closed in $X^\wedge \times Y$ ($X^\wedge$ is the completion of X), and b) for $\epsilon>0$ there is $d>0$ such that if $y=f(x)$, then the closure of $f[V_\epsilon(x)]$ contains $V_d(y)$. Then f is a (uniformly) open map. If further the graph of f is complete, so is the range. *J. L. Kelley* (Berkeley, Calif.).

*Šilov, G. E. **Vvedenie v teoriyu lineĭnyh prostranstv.** [Introduction to the theory of linear spaces.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 384 pp. 7.30 rubles.

This text is designed to carry students from elementary algebra and calculus up to the fundamental ideas of Fourier series, Hilbert spaces, and integral equations. The first eleven chapters, except for an occasional use of the space of continuous functions, are concerned with finite-dimensional spaces. The proofs are given in detail, with many examples and with plenty of exercises for which a student can test his understanding of the material. An outline of the material covered will help to illustrate the author's purpose.

Chapter 1: Systems of linear equations, matrices, determinants, Cramer's rule. Chapter 2: Axioms for a linear space, bases, dimension. Chapter 3: Systems of k equations in n unknowns, rank, dimension of solution set. Chapters 4–6: Linear forms and operators, change of basis, reduction of bilinear or quadratic forms to normal form. Chapters 7–8: The inner product in a Euclidean space is illustrated in N-space and in the space of continuous functions; then distance, orthogonality, and adjoint operations are defined; the Schmidt orthogonalization process is illustrated by defining the Legendre polynomials. Chapter 9: Characteristic numbers and vectors of an operator, symmetric matrices. Chapters 10–11: Reduction of quadratic forms under orthogonal transformations, minimax property of characteristic numbers; quadric surfaces in n-space, full classification with pictures for $n=3$. Chapter 12: Metric spaces and normed spaces, completeness, compactness, norm of an operator. Chapter 13: Fourier series with respect to a closed orthonormal system, trigonometric functions and Legendre polynomials, least square convergence and uniform convergence. Chapter 14: Completely continuous linear operators, their characteristic numbers and vectors, Fredholm integral equations, Sturm-Liouville operators.

For students interested in the applications of mathematics to physics or for mathematics students not yet broken to abstract axiomatic thinking, this course should furnish an excellent transition from the familiar elementary courses to the theory of linear operations. *M. M. Day.*

Ōhira, Keishirō. On some characterizations of abstract Euclidean spaces by properties of orthogonality. Kumamoto J. Sci. Ser. A. 1, no. 1, 23–26 (1952).

This note adds to results of R. C. James [Duke Math. J. 12, 291–302 (1945); Trans. Amer. Math. Soc. 61, 265–292 (1947); these Rev. 6, 273; 9, 42], this reviewer [ibid. 62, 320–337 (1947); these Rev. 9, 192], and E. R. Lorch [Ann. of Math. (2) 49, 523–532 (1948); these Rev. 10, 129] by showing that certain new implications among properties of orthogonality in an inner-product space serve to characterize inner-product spaces among Banach spaces. The paper leaves open this question: Is a space in which Roberts' orthogonality implies Pythagorean orthogonality an inner-product space? An example of a two-dimensional space can easily be constructed in which no two non-zero points are Roberts orthogonal; this settles the question in the negative.

M. M. Day (Urbana, Ill.).

Bonsall, F. F. Core-preserving transformations of a vector space. Proc. Cambridge Philos. Soc. 49, 15–25 (1953).

Generalizing the notion of core due to Knopp, Agnew, and others, the author supposes X to be a complex vector space with a positively homogeneous, subadditive real-valued functional p, and defines the core $K(x)$ of x in X to be the set of all complex α such that the real part of $\lambda\alpha$ is not greater than $p(\lambda x)$ for all complex λ. The central problem is to decide whether a linear operation T of X into X is core-preserving, or whether more generally one has $K(T_1x)\subset K(T_2x)$ for some pair T_1, T_2, and all x. Moreover, the paper generalizes this problem by admitting any Frobenius algebra over the reals rather than merely the complex field, and employs an extension theorem established by Bonsall and Goldie [same Proc. 49, 1–14 (1952); these Rev. 14, 768]. The conditions for core-preservation require the boundedness of T with respect to a pseudo norm $\|\ \|^*$ which depends on p, on the trace-function used in the Frobenius algebra H of scalars, and on the norm in H. This turns out to be the maximum-modulus of elements in $K(x)$. Moreover, X is supposed to contain an element u such that $K(u)=\{1\}$. Then T is core-preserving precisely if $\|Tx\|^*\leq\|x\|^*$ for all x in X. This result is further generalized and applications are given. *R. Arens.*

Feller, William. Semi-groups of transformations in general weak topologies. Ann. of Math. (2) 57, 287–308 (1953).

Let $\{T_t\}$, $0<t<\infty$, be a family of bounded linear operators from a real Banach space X to X such that $T_{t+s}=T_tT_s$. The strong measurability in t of T_t together with $T_0=I$ implies the existence of the infinitesimal generator $Ax=\text{strong }\lim_{h\downarrow 0}h^{-1}(T_hx-x)$ for dense x such that $T_t=\exp(tA)$ [E. Hille and the reviewer]. In view of a wider applicability, the author abandons the hypothesis of the strong measurability. He investigates the adjoint semigroup $\{T_t^*\}$ operating upon the adjoint space X^* of X to X^*: $(T_t^*y,x)=(y,T_tx)$ for $y\in X^*$, $x\in X$ where (y,x) denotes the value at x of the bounded linear functional y. He starts with the (η)-measurability of T_t^*, viz., the measurability in t of (y,T_tx) for $y\in\eta$, $x\in X^*$ where η is a (closed linear) subspace of X^* invariant under T^*. The essential norm $\|T_{t_0}^*y\|_e$ is defined as the infimum of α such that, for each x and each $\epsilon>0$, the t-set $\{t;\ |(y,T_tx)|>(\alpha+\epsilon)\|x\|\}$ has density 0 at t_0. It is proved that

$$\|T_t^*\|_e=\sup_{y\in\eta}\|T_t^*y\|_e/\|y\|$$

is measurable in t and bounded in every closed subinterval of $(0,\infty)$. The subspace of X^* spanned by the Dunford-Pettis integrals $y_{a,b}=\int_a^b T_t^*y\,dt$, $y\in\eta$, is denoted by η_1. Then the subspace $\bar{\eta}$ spanned by η and η_1 is invariant under T_t^* and the operation of the D-P integral such that T_t is (η)-measurable. Moreover, η_1 is invariant by T_t^*, and T_t^* is strongly continuous for $t>0$. The author then defines the equivalence $y_1\sim y_2$ by $T_t^*y_1=T_t^*y_2$ for almost all t. The subspace spanned by η and all the functionals equivalent to some element of η will be denoted by $\eta_{ess\,t}$, and let $\|T_t^*\|_E$ be the essential norm with respect to $\eta_{ess\,t}$. Next let $\eta=\bar{\eta}$. Then $\|T_{t+s}^*\|_E\leq\|T_t^*\|_E\|T_s^*\|_E$ implies the existence of $p>0$ such that $\exp(-pt)\|T_t^*\|_E$ is bounded at ∞. If we denote by H the set of those $y\in\eta$ for which $\|\exp(-pt)T_t^*y\|_E$ is bounded near $t=0$, then the infinitesimal generator Ω of T_t^* in η is defined by $\Omega Y=y'\sim y$ when $y=\text{Banach limit}_{h\downarrow 0}$ of $h^{-1}(T_h^*Y-Y)$ exists for a $Y\in H$. By the consideration of the resolvent of Ω, it is proved that $Y\in\eta_1$ is in the domain of Ω if and only if Y is representable as the Laplace integral $Y=\int_0^\infty\exp(-\lambda t)T_t^*y\,dt$ with $\lambda>p$ and $y\in H$.

K. Yosida (Osaka).

Hille, Einar. On the generation of semi-groups and the theory of conjugate functions. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 122–134 (1952).

This paper appeared earlier in Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **21**, no. 14 (1952); these Rev. **13**, 660.

Phillips, R. S. Perturbation theory for semi-groups of linear operators. Trans. Amer. Math. Soc. **74**, 199–221 (1953).

The author investigates which of the properties of a semigroup are stable. According to his definition a property is stable if it persists when we perturb the infinitesimal generator by a bounded operator. Theorem 3.2 asserts that if A generates the semigroup $T(s)$, $s \in \mathfrak{S}$ a semimodule, $T(s_1+s_2)=T(s_1)T(s_2)$, $s_1, s_2 \in \mathfrak{S}$, $T(0)=I$, $T(s)$ strongly continuous on $\mathfrak{S}$, and B is a bounded operator, then $A+B$ generates a semigroup $S(x)$ satisfying the same conditions. Theorem 3.3 shows that if $\mathfrak{S}=[0, \infty)$ and in addition $\|T(s)\| \leq \exp(\omega s)$, then there exists an ω_1 such that $\|S(s)\| \leq \exp(\omega_1 s)$. According to Theorem 3.5, $S(s)=\sum_0^\infty S_n(s)$, where $S_0(s)=T(s)$ and $S_n(s)x=\int_0^s T(s-\sigma)BS_{n-1}(\sigma)x\,d\sigma$, and for some M,

$$\|S(s)-\sum_0^n S_k(s)\| \leq M(M\|B\|)^{n+1}s^{n+1} \exp(\omega s)/(n+1)!.$$

If $B(z)$ is analytic in a domain of the complex plane, then so is $S(s, z)$. If $T(s)$ is uniformly continuous for $s>0$, so is $S(s)$. A theorem of a type similar to the first is proved if $\mathfrak{S}$ is an angle in the complex plane $\alpha<\arg s<\beta$. The fact that the resolvent of the generator is completely continuous proves to be a stable property. The property that a semigroup be eventually continuous in the uniform topology is not a stable property.

The balance of the paper is devoted to operator differential equations. $T(s)$ satisfies $dT(s)x/ds=AT(s)$. The author investigates $dU(s)x/ds=[A+B(s)]U(s)x$, where $B(s)$ is a strongly continuously differentiable mapping from $[0, \infty)$ into the bounded operators. The solution is $U(s)=\sum_0^\infty S_n(s)$ where $S_0(s)=T(s)$, $S_n(s)=\int_0^s T(s-\sigma)B(\sigma)S_{n-1}(\sigma)d\sigma$. Here $U(s)$ is strongly continuous, strongly continuously differentiable, and unique. A similar theorem is the following. If A is the generator of $T(s)$, $B(s)$, $f(s)$ strongly continuously differentiable mappings of $[0, \infty)$ into the bounded operators and the Banach space respectively, then for each $x \in D(A)$, there exists a continuously differentiable mapping of $[0, \infty)$ into the Banach space $y(s)$ such that

$$y'(s)=[A+B(s)]y(s)+f(s), \quad y(0)=x.$$

It is given by $y(s)=U(s)x+\sum_0^\infty w_n(s)$, where $U(s)$ is as above and

$$w_0(s)=\int_0^s T(s-\sigma)f(\sigma)d\sigma,$$

$$w_n(s)=\int_0^s T(s-\sigma)B(\sigma)w_{n-1}(\sigma)d\sigma.$$

František Wolf (Berkeley, Calif.).

Inaba, Mituo. A theorem on fixed points and its application to the theory of differential equations. Kumamoto J. Sci. Ser. A. 1, no. 1, 13–16 (1952).

Let K be a convex compact subset of a Banach space B, and φ a continuous map of K into itself depending on a parameter σ, where σ denotes points in a second Banach space B_σ. For fixed σ, let M_σ denote the set of fixed points of φ. The main theorem states sufficient conditions for the "continuous" dependence of M_σ on σ. As an application, a theorem concerning the "continuous" dependence of the solutions $x_i=x_i(t)$ $(i=1, 2, \cdots, n)$ of the system of ordinary differential equations

$$(*) \qquad \frac{dx_i}{dt}=f_i(x_1, \cdots, x_n, t) \quad (i=1, 2, \cdots, n)$$

on the initial conditions and on the f_i is proved. This theorem does not suppose that the solution of (*) is unique.

E. H. Rothe (Ann Arbor, Mich.).

Hewitt, Edwin. Linear functionals on almost periodic functions. Trans. Amer. Math. Soc. **74**, 303–322 (1953).

The author starts with an investigation of the normed linear space $\mathfrak{A}^*$ of linear functionals on the normed linear space $\mathfrak{A}$ of ordinary almost periodic (a.p.) functions (with the sup-norm). Besides the representation of $\mathfrak{A}^*$ by help of the countably additive, regular Borel measures on the Bohr compactification of the straight line, he gives a representation of $\mathfrak{A}^*$ by help of the bounded, regular, finitely additive (b.r.f.a.) measures on the smallest algebra $\mathcal{S}^*$ of sets containing the set $\mathcal{S}$ of sets $[x; f(x) \neq 0, f(x)$ a.p.$]$, each measure μ being given a certain norm $|\mu|$. The word "regular" is defined appropriately with regard to $\mathcal{S}$. For a given functional, the measure constructed is defined on a field which is in general larger than that of the measure obtained by applying an analogous construction of Bochner [Ann. of Math. (2) **40**, 769–799 (1939), theorem 1; these Rev. **1**, 110]. An analogue to Bochner's theorem on continuous positive definite (c.p.d.) functions is given: An arbitrary, not necessarily continuous, p.d. function on the straight line is represented in a one-to-one fashion by help of a Fourier-Stieltjes transform of a b.r.f.a., non-negative measure on $\mathcal{S}^*$. A theorem of Yosida and Hewitt [Trans. Amer. Math. Soc. **72**, 46–66 (1952); these Rev. **13**, 543] which indicates a unique splitting of a finitely additive measure (real-valued and bounded) into a countably additive measure and a "purely finitely additive" measure gives rise to a corresponding unique splitting $p=p_c+p_d$ of a p.d. function into a continuous and a purely discontinuous p.d. function (i.e., a p.d. function p_d such that there exists no c.p.d. $q \neq 0$ and p.d. q' with $p_d=q+q'$). Various relations between the two measures, corresponding, in the two above-mentioned representations of $\mathfrak{A}^*$, to a linear functional on $\mathfrak{A}$, are indicated. In this discussion enters a homeomorphic representation of the Bohr compactified line on the (suitably topologized) space of all b.r.f.a. measures on $\mathcal{S}^*$ which assume only the values 0 and 1. There is also a section devoted to the study of p.d. functions on the line and their corresponding b.r.f.a. measures on $\mathcal{S}^*$. *E. Følner.*

Šmul'yan, Yu. L. Isometric operators with infinite deficiency indices and their orthogonal extensions. Doklady Akad. Nauk SSSR (N.S.) **87**, 11–14 (1952). (Russian)

Results announced by Livšic [same Doklady (N.S.) **58**, 13–15 (1947); Mat. Sbornik N.S. **26(68)**, 247–264 (1950); these Rev. **9**, 446; **11**, 669] are extended here. Let H be a Hilbert space, G a closed linear subspace of H, and T a bounded operator on H such that T is isometric on G and such that $f \in G^\perp$ implies $Tf \in (T(G))^\perp$. Let T on G be denoted by V and on $G^\perp$ by W. Any operator L such that $V \subset L \subset T$ is called an orthogonal extension of V. Let

$w(T,\zeta)$ be defined as $(T-\zeta I)(T-\zeta T^*)^{-1}$ on the domain $G^\perp$ for all complex ζ such that $\bar{\zeta}^{-1}$ is a regular point for T. Let $w(\zeta)=-w(V,\zeta)$. A number of theorems are announced, some proofs being sketched, of which the following are typical. 1) T has a bounded inverse if and only if W has a bounded inverse. 2) If $w(T,\zeta)$ exists, then

$$w(T,\zeta)=[W-w(\zeta)][E-W^*w(\zeta)]^{-1},$$

where E is the identity operator on G. 3) Suppose that V has equal deficiency indices, so that V has a unitary extension, and let U be any such extension. Let $w_D(\zeta)$ be the operator $U^{-1}w(\zeta)$. Let ζ_0 be a complex number of absolute value 1 such that for some positive number k, $\|Vf-\zeta_0 f\|\geq k\|f\|$ for all $f\in G$. Then $w_D(\zeta)$ is regular and unitary on an arc of the unit circle containing ζ_0. 4) If $w_D(\zeta)$ is regular and unitary on an arc of the unit circle, then: a) every point of this arc has the property ascribed to ζ_0 in 3) above; b) for every ζ on this arc, V admits a unitary extension for which ζ is a regular point. For the case in which V has finite deficiency indices, this is a theorem of Livšic [second cited article]. *E. Hewitt.*

Kaplansky, Irving. Products of normal operators. Duke Math. J. **20**, 257–260 (1953).

The author gives some conditions, more general than those given by Wiegmann [same J. **15**, 633–638 (1948); **16**, 535–538 (1949); these Rev. **10**, 230; **11**, 153], which are sufficient for the normality of the operators A, B and AB on a Hilbert space to imply the normality of BA. He also gives a Gegenbeispiel to shew that some extra condition is necessary. *A. F. Ruston* (London).

Iséki, Kiyosi. Sur les anneaux normés de Hilbert. I. Sur l'existence d'une projection minimale. C. R. Acad. Sci. Paris **236**, 1123–1125 (1953).

This paper contains several very elementary theorems about rings which are also Hilbert spaces, and in which each element has one or more "adjoints". The author does not appear to be aware of the work of Ambrose on H^*-algebras [Trans. Amer. Math. Soc. **57**, 364–386 (1945); these Rev. **7**, 126]. *A. E. Taylor* (Los Angeles, Calif.).

***Loomis, Lynn H. An introduction to abstract harmonic analysis.** D. Van Nostrand Company, Inc., Toronto-New York-London, 1953. x+190 pp. $5.00.

"The book is an outcome of a course given at Harvard first by G. W. Mackey and later by the author". It gives a self-contained exposition of Gelfand's theory of (commutative) Banach algebras and its application to harmonic analysis on locally compact abelian groups. No special prerequisites are assumed in the readers other than elementary modern algebra and Liouville's theorem from the theory of analytic functions. The contents are as follows. I: Topology. II: Banach spaces. III: Integration. IV: Banach algebras. V: Some special Banach algebras. VI: The Haar integral. VII: Locally compact abelian groups. VIII: Compact groups and almost periodic functions. IX: Some further developments.

Starting with set-theoretical notations, the necessary tools from set theory, topology, Banach spaces and Daniell's theory of integration are given in I–III for the use of later chapters. Next follows a complete exposition of Gelfand's theory of (commutative) Banach algebras with or without the unit. V is devoted to Banach algebras with involution and their application to the formulation, in Banach algebra language, of i) Herglotz-Bochner-Weil-Raikov's representation of the linear positive functional on a semi-simple, self-adjoint, commutative algebra A and of ii) the Plancherel theorem for such A. The latter is formulated by refining a method of Godement [Ann. of Math. (2) **53**, 68–124 (1951); these Rev. **12**, 421]. Thus the book "neglects certain other tools such as the Krein-Milman theorem for the convex sets in the conjugate space of a Banach space and von Neumann's direct integral of Hilbert spaces. "It is believed, however, that the elementary theory of Banach spaces and its application constitutes a portion of the subject which in itself is of general interest and usefulness to the present-day researches". In this way, the theory of "abstract harmonic analysis" is exposed in chapters I–V. The following chapters treat "concrete harmonic analysis on locally compact abelian groups", such as positive definite functions and the generalized Bochner theorem, the Fourier transform and Plancherel theorem, the Wiener Tauberian theorem and the Pontrjagin duality theorem, and finally the theory of almost periodic functions. Thus the book may be considered as a modernization, by means of Banach algebras, of the treatment in A. Weil's "L'intégration dans les groupes topologiques . . ." [Hermann, Paris, 1940; these Rev. **3**, 198].

The author concludes, in X, with a few pages of introduction to the non-commutative theory of unitary operator representations of locally compact groups (Gelfand, Godement, Mackey, Segal, Kaplansky, etc.) and to other problems from the commutative theory (Mackey's Laplace transform, harmonic analysis and synthesis by Beurling, Segal-Kaplansky-Helson, and Schwartz, and finally Levitan-Powzner's Banach algebra treatment of Sturm-Liouville's problem for the second order linear differential equation $y''-(\rho(x)-\lambda)y=0$). *K. Yosida* (Osaka).

Bourgin, D. G. Some multiplicative functionals. Canadian J. Math. **5**, 174–178 (1953).

Poursuivant son étude des fonctionnelles multiplicatives sur une algèbre normée [Proc. Nat. Acad. Sci. U. S. A. **36**, 564–570 (1950); ces Rev. **12**, 421], l'auteur détermine les fonctionnelles multiplicatives continues sur les algèbres suivantes: 1) l'algèbre C^0 des fonctions (réelles ou complexes) définies dans un espace discret, s'annulant hors d'un ensemble dénombrable et tendant vers 0 au point à l'infini, la norme étant le maximum de la valeur absolue; 2) l'algèbre L^p formée des éléments de C^0 dont la somme des puissances p-èmes des valeurs absolues converge, avec la norme usuelle ($p\geq 1$). Par transformation de Fourier, il ramène au second des problèmes précédents la détermination des fonctionnelles multiplicatives continues sur l'algèbre $L^1(I)$ des fonctions de carré sommable dans un intervalle compact I, la multiplication étant la convolution. *J. Dieudonné.*

Kelley, J. L., and Vaught, R. L. The positive cone in Banach algebras. Trans. Amer. Math. Soc. **74**, 44–55 (1953).

Lemmas are proved on ordered Banach spaces R such that the non-negative cone C contains an element e where $\|e\|=1$ and $\{y\mid\|e-y\|\leq 1\}\subset C$. For example, in such a space (a) $\|f\|=f(e)$ for $f\in C'$, the cone polar to C, and (b) dist $(-x,C)=\sup\{f(x)\mid f\in C' \text{ and } \|f\|\leq 1\}$. It is then observed that if a Banach algebra is ordered via a positive cone which is the closure of the set of sums of squares (sums of elements xx^* when ordering self-adjoint elements of *-algebras), the hypotheses of these lemmas apply. By this means in a systematic manner a number of known and new theorems are proved. In the latter category is the following: let $f(t)=\sum_{n=-\infty}^{\infty}a_n e^{int}$ be absolutely convergent and real-

valued and let S be the class of all (two-ended sequences) which are zero except for a finite number of terms with non-negative indices; then

$$\max\left[\sup\left\{f(t)\,\middle|\,0\leqq t<2\pi\right\},0\right]=\inf\left\{\sum_{n=-\infty}^{\infty}|a_n+\sum_{m=-\infty}^{\infty}b_m\delta_{m-n}|\,\middle|\,b\in S\right\}.$$

S. Sherman (Sherman Oaks, Calif.).

Fukamiya, Masanori. On a theorem of Gelfand and Neumark and the B^*-algebra. Kumamoto J. Sci. Ser. A. 1, no. 1, 17–22 (1952).

Let R be a B^*-algebra with an identity element e. Let E be the set of all self-adjoint elements of R, and let D (D_0) be the set of all elements of E with non-negative (positive) spectra. Let P be the set of real-valued linear functionals on E which are non-negative on D, and let $\mathfrak{P}$ be the set of positive functionals on R (i.e., the set of f such that $f(x^*x)\geqq 0$ for all $x\in R$). Then D and D_0 are convex cones and D_0 is the interior of D. If H is a closed subspace of E such that $H\cap D$ is empty, then there is a nonzero $f\in P$ such that $f(H)=0$. There is a nonzero element in $\mathfrak{P}$ if and only if $x^*x+y^*y+\cdots+z^*z=-e$ is impossible. The following conditions are equivalent: R is a C^*-algebra; $x^*x+y^*y+\cdots+z^*z=0$ implies $x=y=\cdots=z=0$; $P=\mathfrak{P}$ (in the obvious sense). [Reviewer's remarks: That D is a convex cone has been proved independently by Kelley and Vaught [Theorem 4.7 of the paper reviewed above]. As Kaplansky has observed, the convexity of D has as a consequence an affirmative answer to the conjecture that every B^*-algebra is a C^*-algebra [Gelfand and Neumark, Mat. Sbornik N.S. 12(54), 197–213 (1943), p. 198; these Rev. 5, 147]. The proof is based on the known result that xy and yx always have the same spectrum. In fact, if $-x^*x\in D$ and $x=u-iv$ with $u,v\in\mathfrak{E}$, then $x=0$ since $2u^3+2v^3-x^*x-xx^*=0$ and each summand is in the convex cone D. It follows from this that $y^*y\in D$ for every y since for $y^*y=u-v$, with $u,v\in D$ and $uv=0$, we have $(yv)^*yv=-v^3$ and therefore $v=0$. The case of a B^*-algebra without an identity has been handled by Kaplansky by embedding in an algebra with an identity and by Rickart by direct computation (both unpublished).] *J. A. Schatz* (Bethlehem, Pa.).

Šilov, G. E. On decomposition of a commutative normed ring in a direct sum of ideals. Mat. Sbornik N.S. 32(74), 353–364 (1953). (Russian)

Let R be a commutative Banach algebra with unit element, X its space of maximal ideals. If R is a direct sum, it is trivial that X is disconnected in the appropriate way. This paper is concerned with the converse. With the additional assumption that R admits a suitable *-operation, an affirmative answer was already given in Gelfand's original paper [Mat. Sbornik N.S. 9(51), 3–24 (1941); these Rev. 3, 51], but the general question has remained open till now.

The key idea can be explained as follows. If there exists an element with a disconnected spectrum, then by a well-known application of the Cauchy integral it is possible to construct an idempotent. However, the author abandons the search for a single element of this kind. Instead he argues successfully that there must exist a finite subset $x_1,\cdots,x_n$ whose joint spectrum (i.e., the set of all points $x_1(M),\cdots,x_n(M)$ in n-space) is disconnected in the right way. An application of the n-dimensional Cauchy integral then does the trick. Requisite facts from the theory of several complex variables are quoted from the book of Fuks [Theory of analytic functions of several complex variables, OGIZ, Moscow-Leningrad, 1948; these Rev. 12, 328].

I. Kaplansky (Chicago, Ill.).

Šilov, G. E. On rings of functions with uniform convergence. Ukrain. Mat. Žurnal 3, 404–411 (1951). (Russian)

The author first establishes a simple generalization of the complex form of the Stone-Weierstrass theorem [M. H. Stone, Math. Mag. 21, 167–184, 237–254 (1948); these Rev. 10, 255], as follows. Let $C(G)$ be the (complex) Banach algebra of all continuous complex-valued functions on the compact Hausdorff space G, with the usual algebraic operations and norm. Let L be a closed subalgebra of $C(G)$ containing all constants. Let Λ be a closed subalgebra of L such that $x\in\Lambda$ implies $\bar{x}\in\Lambda$. The equivalence relation $\sim$ on G such that $t_1\sim t_2$ if and only if $f(t_1)=f(t_2)$ for all $f\in\Lambda$ obviously dissects G into disjoint closed sets τ. For every such τ, let $J(\tau)$ be the ideal in L of all functions in L which vanish on τ. There is an obvious and natural isomorphism carrying the difference algebra $L-J(\tau)$ onto an algebra of functions defined on τ. The generalized Stone-Weierstrass theorem asserts that if $f\in C(G)$ and if f agrees on every τ with a function in $L-J(\tau)$, then $f\in L$. For $L=\Lambda=C(G)$, this is exactly the Stone-Weierstrass theorem. The theorem is applied to prove the following result. Let C be the algebra $C(|z|\leqq 1)$ and A the closed subalgebra of C consisting of the functions which are analytic on $|z|<1$. Let $\{A,\Sigma\}$ be the smallest closed subalgebra of C containing A and the real functions $f\in\Sigma$. A closed subset S of $|z|\leqq 1$ is said to be admissible if S has void interior and if for all z_0 non-$\in S$, $|z_0|<1$, there is a continuous curve running from z_0 to $|z|=1$ which does not intersect S. Then it is proved that $\{A,\Sigma\}=C$ if and only if all sets of points equivalent under the set of functions Σ are admissible. This generalizes a theorem attributed to Hurgin [Moskov. Gos. Univ. Učenye Zapiski 145, Ser. Mat. 3 (1949) (unavailable)].

E. Hewitt (Seattle, Wash.).

Berezanskiĭ, Yu. M. On certain normed rings constructed from orthogonal polynomials. Ukrain. Mat. Žurnal 3, 412–432 (1951). (Russian).

This paper deals with functions expansible in absolutely convergent series of orthogonal polynomials: Čebyšev, Legendre, Jacobi, and others. A certain part of the discussion is cast in a very general form, as follows. Let Q be a compact subset of $(-\infty,\infty)$ and let σ be a non-negative countably additive Borel measure on Q. Let $\{P_n(t)\}_{n=0}^{\infty}$ be the set of polynomials on Q orthonormalized with respect to σ (P_n has degree n). Let $0<\mu_0\leqq\mu_1\leqq\mu_2\leqq\cdots\leqq\mu_n\leqq\cdots$ be a sequence of numbers such that $|P_n(t)|\mu_n^{-1}$ is uniformly bounded for all n and all $t\in Q$. Let $\Lambda(d\sigma,\mu)$ be the space of all complex functions $x(t)$ on Q such that $x(t)=\sum_{j=0}^{\infty}x_jP_j(t)$ and also $\|x\|=\sum_{j=0}^{\infty}|x_j|\mu_j<\infty$. With pointwise algebraic operations and the above norm, $\Lambda(d\sigma,\mu)$ is obviously a complex Banach space, and under certain conditions it is also a Banach algebra, with $\|xy\|\leqq\|x\|\cdot\|y\|$. The space $\Lambda(d\sigma,\mu)$ is of course commutative and has a unit whenever it is a Banach algebra. A simple but not easily applied necessary and sufficient condition that $\Lambda(d\sigma,\mu)$ be a Banach algebra is that $\|P_jP_k\|\leqq\mu_j\mu_k$ for all j and k. The maximal ideal space $\mathfrak{M}$ of $\Lambda(d\sigma,\mu)$ is a closed subset of the complex plane contained in $|z|\leqq\|z\|$ and containing Q, the homomorphisms of $\Lambda(d\sigma,\mu)$ onto the complex number field all being of the form $\sum_{j=0}^{\infty}x_jP_j(t)\sim\sum_{j=0}^{\infty}x_jP_j(z)$, where $z\in\mathfrak{M}$. The real z's

in $\mathfrak{M}$ are exactly the symmetric maximal ideals. It is proved that all maximal ideals are symmetric (i.e., $\mathfrak{M}$ consists entirely of real numbers) if all μ_n are bounded. A number of examples are considered next. Q is taken to be $[-1, 1]$, and $d\sigma(t)$ is first taken as $(1-t^2)^{m/2-1}dt$ $(m=1, 2, 3, \cdots)$, with μ_n a rather complicated function of n and m. The spaces $\Lambda((1-t^2)^{m/2-1}dt, \mu)$ so obtained correspond to the spaces of functions expansible in series of zonal functions for m-dimensional rotation groups [see M. G. Kreĭn, same Žurnal **1**, no. 4, 64–98 (1949); **2**, no. 1, 10–59 (1950); these Rev. **14**, 480; **12**, 719; **13**, 1139], and all of these spaces are Banach algebras, with $\mathfrak{M}=[-1, 1]$ in every case. The space

$$\Lambda((1-t)^\alpha(1+t)^\beta dt, \quad \mu_j=K(2j+1)^A),$$

where

$$-1<\alpha, \beta\leqq -\tfrac{1}{2}, \quad A\geqq 1,$$

and K is a certain constant, is shown to be a Banach algebra. (The polynomials P_n are here the Jacobi polynomials $P_n^{\alpha,\beta}$, of course.) A theorem of Gel'fand is applied to prove an analogue of Wiener's theorem on functions with absolutely convergent Fourier series. A special case of the theorem is the following. If $x(t)=\sum_{j=0}^{\infty}x_jP_j^{\alpha,\beta}(t)$ and if $\sum_{j=1}^{\infty}|x_j|j^A<\infty$, where $A\geqq 1$, then $f(x(t))$ has an expansion with the same properties, f being any function analytic on the set of values of $x(t)$, as t goes from -1 to 1. Given two spaces $\Lambda(d\sigma, \mu)$ and $\Lambda(d\tilde{\sigma}, \tilde{\mu})$ on the same set Q, one defines the transformation operator U carrying $\Lambda(d\sigma, \mu)$ into $\Lambda(d\tilde{\sigma}, \tilde{\mu})$ by the relations $y_j=(Ux)_j=\int_Q x(t)\overline{P}_j(t)d\tilde{\sigma}(t)$. That is, one simply expands the function $x(t)$ in a series of the polynomials $\overline{P}_j$. The inverse V of U is defined analogously. No general discussion is given for the existence of U or V. A complicated theorem is proved giving sufficient conditions that U and V be defined and continuous on the spaces $\Lambda(p(t)dt, \mu)$ and $\Lambda(p(t)h(t)dt, \mu)$. This theorem is applied to show that U and V are both continuous for $\Lambda((1-t^2)^{m/2-1}dt, n^{(m-1)/2})$ and $\Lambda(h(t)(1-t^2)^{m/2-1}dt, n^{(m-1)/2})$ (here $Q=[-1, 1]$), provided that the function $h(t)$ has a bounded derivative of order $[\frac{1}{2}(m+5)]$, for $m>1$, and of order 2 for $m=1$. Some of the results of this paper have been announced earlier [Doklady Akad. Nauk SSSR (N.S.) **81**, 329–332 (1951); these Rev. **13**, 952]. *E. Hewitt* (Seattle, Wash.).

Calculus of Variations

Kerimov, M. K. On two-dimensional discontinuous problems of the calculus of variations. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze **18**, 209–219 (1951). (Russian. Georgian summary)

The author considers the two-dimensional non-parametric minimum problem defined by the integral

$$I(z)=\iint_G F(x, y, z, p, q)dxdy,$$

where G is the region interior to a simple closed curve S on which the values of $z(x, y)$ are prescribed. As admissible functions he takes continuous functions with first partial derivatives continuous except for discontinuities of the first kind across a smooth curve σ which cuts G into two regions. He finds conditions, analogous to the one-dimensional Weierstrass-Erdmann corner conditions, which must be satisfied by $z(x, y)$ across σ. He considers also the case of discontinuous F. *J. M. Danskin* (Washington, D. C.).

Finn, Robert. A property of minimal surfaces. Proc. Nat. Acad. Sci. U. S. A. **39**, 197–201 (1953).

The author establishes the following property of single-valued solutions $\varphi(x, y)$ of the second-order elliptic partial differential equation of minimal surfaces, analogous to a known property of harmonic functions: Let D be a bounded open set of the plane, and let d and $\mathfrak{A}$ be positive variables. It is shown that there exists a positive function $K_D(\mathfrak{A}, d)$, nondecreasing in $\mathfrak{A}$ for every fixed d, and nonincreasing in d for every fixed $\mathfrak{A}$, such that if $\varphi(x, y)$ is defined in D, the area of the surface represented by $z=\varphi(x, y)$ is not more than $\mathfrak{A}$, and P is a point of D at distance not less than d from the boundary of D, then $\varphi_x^2+\varphi_y^2\leqq K_D(\mathfrak{A}, d)$ at P. Generalizations are made to bounded minimal surfaces and to a well-defined class of quasi-linear elliptic partial differential equations, and applications are made to existence and uniqueness theorems for boundary value problems.

E. F. Beckenbach (Los Angeles, Calif.).

Heinz, Erhard. Über die Lösungen der Minimalflächengleichung. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1952**, 51–56 (1952).

For brevity, the symbol (E) will be used to refer to the partial differential equation $(1+q^2)r-2pqs+(1+p^2)t=0$ of minimal surfaces in non-parametric form. According to a classical result of S. Bernstein, the linear functions $ax+by+c$ are the only solutions of (E) which are of class C'' in the whole xy-plane. This remarkable result appears as an immediate consequence of the following inequalities established in the present paper. Let $z(x, y)$ be a solution of (E) which is of class C'' in the circular disc with center (x_0, y_0) and radius R_0, and let p_0, q_0, r_0, s_0, t_0 denote the values at (x_0, y_0) of the first and second partial derivatives of $z(x, y)$. Then $r_0^2+s_0^2$ and $s_0^2+t_0^2$ are both less than or equal to $(2\pi^2/3R_0^2)(2+p_0^2+q_0^2)(1+p_0^2)^2$. The proof is made by reducing these inequalities to certain simple inequalities for harmonic functions, by the use of Riemann's theorem that any family of parallel planes intersects a minimal surface in an isothermic family of curves. *T. Radó.*

Bellman, Richard, Glicksberg, Irving, and Gross, Oliver. On some variational problems occurring in the theory of dynamic programming. Proc. Nat. Acad. Sci. U. S. A. **39**, 298–301 (1953).

The authors are concerned with the cost of controlling dynamical systems governed by the equation

$$(1) \qquad \frac{dx(t)}{dt}=Ax(t)+f(t), \quad x(0)=c, \quad 0\leqq t\leqq T,$$

where $x(t)$ and $f(t)$ are column vectors and A a constant matrix. The cost is the sum of two terms: the cost of employing the forcing term $f(t)$, measured as a functional on f, and the cost of deviation of the system from the desired state $y(t)$, measured as a functional on $x-y$. Two classes of problems are considered. (i) Minimize the cost of control. (ii) For a a given cost of the use of the forcing term, maximize the output of the system. An example of (i) is given by the following theorem, which is stated without proof. Theorem. Let $x(t)$ be the solution of (1) regarded as a scalar equation with $A=-1$ and $c=1$. Let f be constrained as follows: $0\leqq f(t)\leqq a_1>1$; $\int_0^T f(t)dt\leqq a_2<T$. Then the f which minimizes the integral $\int_0^T(1-x(t))^2dt$ is given by: $f(t)=a_1$ for $a_4\leqq t<a_5$, where $0<a_4<a_5<T$, and $f(t)=0$ elsewhere. The authors present various other results for linear and quadratic cost functionals. The principal difficulty in these

problems lies in the restricted nature of the variations which may be made on $f(t)$. *J. M. Danskin.*

Theory of Probability

***Tornier, Erhard, und Domizlaff, Hans. Theorie der Versuchsvorschriften der Wahrscheinlichkeitsrechnung.** W. Kohlhammer Verlag, Stuttgart, 1952. 108 pp. DM 12.00.

This axiomatization of the theory of probability is designed as an improvement on previous work by one of the authors [Tornier, Acta Math. **60**, 239–380 (1933); Wahrscheinlichkeitsrechnung und allgemeine Integrationstheorie, Teubner, Leipzig-Berlin, 1936]. The inadequacy of these papers, which is said to have been overcome in the present paper, is that "the possible transformations from one prescription for trials (Versuchsvorschrift) to another were not systematically clarified and consequently a unified development of the fundamental problems of the calculus of probability met with difficulties." In the present axiom system probability appears as a finitely additive set function, so that the theory is not one of measure but of content. It is stressed that this theory operates exclusively in the finite domain and that every calculated probability can be verified by comparison with a relative frequency in a sufficiently long sequence of trials. The axiom system does not lend itself to ready summary. *J. Wolfowitz* (Ithaca, N. Y.).

Mittmann, Otfrid M. J. Zur Approximation einer Wahrscheinlichkeitsverteilung mit Hilfe ihrer niederen Momente. Z. Angew. Math. Mech. **33**, 61–63 (1953).

Let $H(x)$ be some (conveniently chosen) density function, defined in $(-\infty, \infty)$, and let $W_0(x), W_1(x), \cdots, W_n(x)$ be polynomials of degrees $0, 1, \cdots, n$ that are orthogonal relative to $H(x)$. The author has worked out formulae for the coefficients $a_0, a_1, \cdots, a_n$ in

$$W(x)=H(x)\{a_0W_0(x)+a_1W_1(x)+\cdots+a_nW_n(x)\}$$

that fulfil the conditions:

$$\text{(i)} \qquad \int_a^b x^k W(x)dx=M_k \quad (k=0, 1, \cdots, m),$$

where $m<n$; (ii) $W(x_j)=y_j$ $(j=1, \cdots, l)$, where $l\leqq n-m$; and when $l<n-m$, (iii) $\int_c^d\{W(x)\}^2dx$ is a minimum subject to (i) and (ii); here $a, b, c, d, M_k, x_j, y_j$ are given constants. His formulae are given explicitly for the cases $l=0, 1, 2$. It is indicated that the interval (c, d) may be replaced by, say, a finite set of intervals. For example, instead of (c, d) we may take the complement of (a, b). *H. P. Mulholland.*

Adam, Adolf. Klassenstatistik. Mitteilungsblatt Math. Statist. **5**, 1–28 (1953).

Scattered elements of probability theory, amply covered by, say, Chs. 2, 6, and 14 of Feller's treatise [An introduction to probability theory . . . , v. 1, Wiley, New York, 1950; these Rev. **12**, 424]. *H. Wold* (Uppsala).

Rényi, Alfred. Sur un théorème général de probabilité. Ann. Inst. Fourier Grenoble **1** (1949), 43–52 (1950).

A review and slight extension of results previously reviewed [J. Math. Pures Appl. (9) **28**, 137–149 (1949); Compositio Math. **8**, 68–75 (1950); Časopis Pěst. Mat. Fys. **74** (1949), 167–175 (1950); these Rev. **11**, 161, 581; **12**, 590]. No number-theoretic applications are given in this paper. *M. Kac* (Ithaca, N. Y.).

Onoyama, Takuji. A representation of a family of random variables and their means. Mem. Fac. Sci. Kyūsyū Univ. A. **6**, 179–183 (1952).

In this representation, a set of random variables is a complete modular lattice, a distribution is a modular function on the lattice, a specific random variable is a specified linear functional defined on a specific ring of operators generated by a set of projections lattice isomorphic to the lattice. *J. L. Doob* (Urbana, Ill.).

Bochner, S. Length of random paths on general homogeneous spaces. Ann. of Math. (2) **57**, 309–313 (1953).

Let M be a metric space with the distance $\rho(x, y)$ and a Lebesgue measure element dv_x for which open sets are measurable. Consider a Markoff chain density $(f(r, x; s, y)$ $(t'<r<s<\infty, t'<0)$ defined in M and which is spatially homogeneous, viz., $f(r, Tx; s, Ty)=f(r, x; s, y)$ and $dv_{Tx}=dv_x$ for a transitive group of isometric transformations T of M on M. Let Λ be the space of all (non-continuous) paths $\lambda=x(t)$, $0\leqq t\leqq 1$, on M through $x_0=x(0)$. Starting with the probability element $\int\cdots\int\prod_{\nu=0}^{n-1}f(t_\nu, x_\nu; t_{\nu+1}, x_{\nu+1})dx_{\nu+1}$ for variable vertices $(x_1, \cdots, x_n)$ and for any fixed partition $\delta=\{0=t_0<t_1<\cdots<t_n=1\}$ of the time interval, a probability measure $d\lambda$ on the space Λ is introduced. The approximate length $F_\delta(\lambda)=\sum_{\nu=0}^{n-1}\rho(x_\nu; x_{\nu+1})$ thus satisfies

$$\Phi_\delta(u)=\int \exp\,(-uF_\delta(\lambda))d\lambda=\prod_{\nu=0}^{n-1}A(t_\nu, t_{\nu+1}; u),$$

where $A(r, s, u)=\int \exp\,(-u\rho(x, y))f(r, x; s, y)dv_y$. If the process is stationary (temporally homogeneous), it is proved that $\Phi(u)$, $=$the Moore-Smith $\lim_{\delta\to 0}\Phi_\delta(u)$, exists and $\Phi(u)=\lim_{\epsilon\to 0}A(0, \epsilon; u)^{1/\epsilon}=\exp\,(-D(u))$, where

$$0\leqq D(u)=\lim_{\epsilon\to 0}\epsilon^{-1}(1-A(0, \epsilon; u)).$$

The value $\Phi(+0)$ is the probability measure of the set of paths whose lengths are finite [cf. the author, Ann. of Math. (2) **48**, 1014–1061 (1947); these Rev. **9**, 193]. The result is applied to the temporally and spatially homogeneous process $f(t; x)=f(r, a; r+t, a+x)$ in Euclidean space E_k which is infinitely subdivisible in the sense that

$$\int \exp\left(i\sum_{j=1}^{k}x_j\alpha_j\right)f(t; x_1, \cdots, x_k)dx_1\cdots dx_k$$
$$=\exp\,(-t\psi(\alpha_1, \cdots, \alpha_k)),$$

where $\psi(0)=0$, $0\leqq \mathrm{Re}\,\psi(\alpha_j)<\infty$. Thus Theorem 3 gives the criterion $\int(1+\sum_j\alpha_j^2)^{-(k+1)/2}\psi(\alpha_1, \cdots, \alpha_k)d\alpha_1\cdots d\alpha_k=\infty$ or $<\infty$ according as almost all paths have infinite lengths or finite lengths. Extensions to the non-stationary case and to non-Euclidean space M are also discussed. *K. Yosida.*

Hodges, J. L., Jr., and Rosenblatt, M. Recurrence-time moments in random walks. Pacific J. Math. **3**, 127–136 (1953).

The authors study Markov chains with states $E_1, E_2, \cdots$ such that transition between any two states occurs with positive probability and such that the recurrence time for some state, and hence any state, is finite with probability one. Let R_j be the recurrence time for the state j and $I(R_j)$ be the largest k such that the kth moment of R_j is finite. It is proved that $I(R_j)$ has the same value for all j and examples of random walk chains are given with first k recurrence time moments finite and higher moments infinite, for any k.

J. L. Snell (Princeton, N. J.).

Bartlett, M. S. Recurrence and first passage times. Proc. Cambridge Philos. Soc. **49**, 263–275 (1953).

Recurrence and first-passage times for time-homogeneous stochastic processes are considered. The author is "more concerned . . . with general structural relations than with specific points of rigour, or specific applications." Let S be a state whose occurrence makes the subsequent behavior independent of the past and let e_n be the probability that S occurs at time n given S at time 0, $n=1, 2, \cdots$. Starting with the known relation between the generating function for the recurrence time of S and $\pi(z)=\sum_1^\infty e_n z^n$, various formulas are developed for the generating functions of first-passage times from S to a different state T either without or irrespective of intermediate recurrence of S; analogous formulas are given for continuous time and discrete or continuous states; these are related to the ergodic properties of Markov processes. General stationary ergodic processes are discussed using the methods of Smoluchowski [Akad. Wiss. Wien, S.-B. IIa. **124**, 339–368 (1915)]. An alternative matrix method is given for finite Markov chains. A special treatment is given of stationary processes having a derivative.

T. E. Harris (Santa Monica, Calif.).

Siegert, Arnold J. F. On the first passage time probability problem. Physical Rev. (2) **81**, 617–623 (1951).

Kern, Enrique Roberto. The theory of risk and its use. Contributions to the study of stability and retention limits in insurance. Univ. Buenos Aires. Rev. Fac. Ci. Econ. **5**, 7–57 (1952). (Spanish)

An expository account with insufficient attention paid to the analogous problems in the general theory of stochastic processes. A three-significant-figure table of

$$\varphi(u, \lambda, \eta)=\frac{1}{1+\lambda}e^{-2\lambda u/\eta^2\lambda}$$

is appended:

$u=25(25)150(50)300(100)400,$
$\lambda=2\text{-}1/2(2\text{-}1/2)20(5)40(10)50,$
$\eta=2(1)10(2\text{-}1/2)15(5)20.$

H. L. Seal (New York, N. Y.).

Kalkanes, Ioannis. On life insurance with unequal insured capital. Bull. Soc. Math. Grèce **27**, 92–110 (1953). (Greek)

Barrett, F. Dermot, and Shepard, Herbert A. A bibliography of cybernetics. Proc. Amer. Acad. Arts Sci. **80**, 204–222 (1953).

Mathematical Statistics

*Smith, Ed Sinclair. **Binomial, normal and Poisson probabilities.** Published by the author, Bel Air, Md., 1953. 71 pp. $2.50.

Author's summary: For anyone who even occasionally needs values, this book has five-place tables of binomial, normal and Poisson probabilities. It includes methods and graphs for readily obtaining values within 0.001, of the binomial cumulative probability for higher n, i.e. over 20, than in the table. These approximations use two terms of the Gram-Charlier series, types A and B, with values from the normal and Poisson tables. This work presents a new three-term modification of the series A, which makes it possible for this accuracy to be attained with the instant self-contained and compact paper. Introductory and background material is also given, along with numerous examples and references.

The paper also contains tables from which more highly precise values of these probabilities can be computed: 10-place tables of $\log n$, e^{-n} for n through 100, and $n!$ and $\log n!$ for n through 200; and 5-place tables of $\binom{n}{r}$ and $\log\binom{n}{r}$ for n through 50. It consists of 71 pages, including 10 tables and 17 graphs. *H. Chernoff* (Stanford, Calif.).

Hoeffding, Wassily. On the distribution of the expected values of the order statistics. Ann. Math. Statistics **24**, 93–100 (1953).

The main result of the paper is the following. Let $X_1, \cdots, X_n$ be n mutually independent random variables with a common cumulative distribution function $F(x)$; let $Z_{n1}\leqq\cdots\leqq Z_{nn}$ be the ordered values $X_1, \cdots, X_n$; and let EZ_{nj} be the expected value of Z_{nj}. Theorem: If $E[|x|]<\infty$ and if $g(x)$ is a real-valued, continuous function such that $|g(x)|<h(x)$, where the function $h(x)$ is convex and $E[h(x)]<\infty$, then $\lim_{n\to\infty} n^{-1}\sum_{j=1}^n g(EZ_{nj})=E[g(x)]$. The distribution of the expected values $E[f(Z_{nm})]$ for certain functions $f(x)$ is also considered. *R. P. Peterson.*

Springer, Melvin D. Joint sampling distribution of the mean and standard deviation for probability density functions of doubly infinite range. Ann. Math. Statistics **24**, 118–122 (1953).

The joint sampling distribution of $\bar{x}$ and S, which is often referred to as Helmert's distribution for the normal population, is derived in this paper in terms of quadratures for any sample size when $f(x)$ is any probability density function of doubly infinite range. The method adopted here is to employ a transformation for fixed n and to derive the limits of integration inductively, and is different from the procedures due to W. Kruskal [Amer. Math. Monthly **53**, 435–438 (1946); these Rev. **8**, 161] and L. Truksa [Biometrika **31**, 256–271 (1940); these Rev. **2**, 109].

T. Kitagawa (Calcutta).

*Cochran, William G. **Sampling techniques.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1953. xiv+330 pp. $6.50.

As stated in the preface "the purpose of the book is to present a reasonably comprehensive account of sampling theory as it has been developed for use in sample surveys, with sufficient illustrations to show how the theory is applied in practice, and with a supply of exercises to be worked by the student." The main theme is the optimum design, under cost considerations, of sample surveys for obtaining point estimates of means (or totals) of finite populations. However, a distinction is made between descriptive and analytic surveys, and the author shows repeatedly that statistical analysis of survey data can yield such knowledge about time invariant properties of a population as can be used for improving the design of a later survey. The chapter headings are: 1. Introduction. 2. Simple random sampling. 3. Sampling for proportions and percentages. 4. The estimation of sample size. 5. Stratified random sampling. 6. Ratio estimates. 7. Regression estimates. 8. Systematic sampling. 9. Type of sampling unit. 10. Subsampling with units of equal size. 11. Subsampling with units of unequal size. 12. Double sampling. 13. Sources of errors in surveys. The book ends with a list of answers to exercises, an author

index, and a subject index. The basic properties of the methods described are stated in about 50 theorems, which are proved by elementary algebra and calculus. (The standard method of seeking for relative extreme values is used repeatedly to determine absolute extreme values without checks being made.) The book is remarkably up to date, more than half of the 96 papers and books referred to having appeared during the last 6 years. The following is a sample of the many particular topics discussed: the effects of bias and non-normality on probability statements, inverse sampling for estimating proportions, the two-sample method for obtaining confidence intervals of prescribed maximal lengths, optimum stratification, the limiting distribution of a function of sample moments, two-dimensional systematic sampling, sampling with unequal probabilities, and sampling on successive occasions with partial replacement. Being well-planned and easy to read, the book is very suitable both as a text and as a handbook. (On p. 33 the word "continuous" and on p. 123 the sentence "the e_i are independent of the x_i" should be deleted. On p. 270 $n'W_h(1-W_h)$ should be replaced by $W_h(1-W_h)/n'$.)

D. M. Sandelius (Uppsala).

Mahalanobis, P. C. Some aspects of the design of sample surveys. Sankhyā 12, 1–7 (1952).

***Liu, Nanming. A new method for computing the mean difference.** Essays and papers in memory of late president Fu SSu-Nien, pp. 339–356. The National Taiwan University, Taipei, 1952. (Chinese)

Expository paper.

Goffman, Casper. Note on the variation of means. Ann. Math. Statistics 24, 307–309 (1953).

The author is interested in the comparison of batch-to-batch variations for different runs in a manufactured product. He assumes, for the simplest case, that the items are distributed normally with the same variance but with means $\mu_1, \mu_2, \cdots, \mu_n$ and obtains under the assumptions of four conditions too long to state here that a measure of the variation among the means μ_i is given by $V=\sum_{i=1}^{n}(\mu_i-\bar{\mu})^2$, where $\bar{\mu}=\sum_{i=1}^{n}\mu_i/n$. From the point of view of quality control the reviewer notes that such a system is out of control and concludes that steps should be taken to eliminate assignable causes leading to such variations in the means. *L. A. Aroian* (Culver City, Calif.).

Goodman, Leo A. A simple method for improving some estimators. Ann. Math. Statistics 24, 114–117 (1953).

Let Y be any real-valued random variable whose distribution depends upon an unknown positive parameter θ and such that $\theta E\{Y|\theta\}/E\{Y^2|\theta\}=A$ identically in θ, where A is known. Then among all statistics of the form aY where a is a constant, the following two properties hold: (1) the risk $R_{aY}(\theta)=E\{\lambda(\theta)(aY-\theta)^2|\theta\}$ is minimized for each value of θ when $a=A$; (2) AY is the only one which is unbiased in the sense of Lehmann with respect to the loss function $W[\hat{\theta}, Y]$ when $\lambda(\hat{\theta})=\hat{\theta}^{-4}$. As applications the statistics from a normal population with unknown mean and variance

$$\sum_{i=1}^{n}(x_i-\bar{x})^2/(n+1)=s^2(n-1)/(n+1)$$

and

$$s=[2(n-1)]^{\frac{1}{2}}\Gamma(n/2)/\Gamma((n-1)/2)$$

are found to be the invariant estimates of σ^2 and σ respectively which minimize the risk. Similarly, for a sample of n from a uniform distribution on $(0, p)$,

$$\max(x_1, x_2, \cdots, x_n)(n+2)/(n+1)$$

is found to be the invariant estimate $\hat{p}$ which minimizes the risk. *T. Kitagawa* (Calcutta).

Bahadur, R. R. A property of the t-statistic. Sankhyā 12, 79–88 (1952).

It is the object of this paper to describe a formulation of the statistical problem known as "testing the significance of two means", and to show that the classical procedure based on the two-sided t test possesses an optimum property in terms of this formulation. It seems to us that as compared to other formulations which have been considered in the literature the present one is more nearly relevant to many applications, so that the result obtained here provides a justification for the manner in which the two-sided t test is widely used in practice. (From author's introduction.)

L. J. Savage (Chicago, Ill.).

Izaki, Mamoru. Convergence of integral and its applications. Bull. Math. Statist. 5, 31–34 (1952).

The author applies the notion of random integration [T. Kitagawa, same Bull. 4, 15–21 (1950); these Rev. 14, 457] to show that a certain statistical calculator gives an unbiased estimate in calculating integrals. He also gives estimates for the variance. *R. H. Cameron.*

Uranisi, Hisao. On the statistical inferences in finite populations by two sample theory. Bull. Math. Statist. 5, 9–20 (1952).

The author extends some results of T. Kitagawa [Mem. Fac. Sci. Kyūsyū Univ. A. 6, 55–95 (1951); these Rev. 14, 390]. These results deal with inferences concerning a sample only part of which is observed. The extension replaces the unknown normal population from which the sample is drawn by one having a Gram-Charlier Type A distribution.

H. Chernoff (Stanford, Calif.).

Terpstra, T. J. A non-parametric k sample test and its connection with the H-test. Math. Centrum Amsterdam. Rapport S 92 (VP 2), 11 pp. (1952).

For testing the hypothesis H_0 that $(X_{i1}, \cdots, X_{in_i})$, $i=1, \cdots, k$, are k independent random samples from the same continuous distribution the author proposes a test based on the statistic

$$T^2=12\sum_{i<j}\frac{1}{n_in_j}\tilde{U}^2_{ij}-\frac{12}{N+1}\sum\frac{1}{n_i}\tilde{U}_i^2,$$

where $N=\sum n_i$, $\tilde{U}_i=\sum_{j=1}^{i-1}\tilde{U}_{ji}-\sum_{j=i+1}^{k}\tilde{U}_{ij}$, $\tilde{U}_{ij}=U_{ij}-n_in_j/2$ and U_{ij} is the number of pairs (ξ, η) with $X_{i\xi}>X_{j\eta}$. When H_0 is true and the n_i tend to infinity in a suitable way, the joint distribution of the U_{ij} is found, by the method of moments, to be asymptotically normal and that of T^2 asymptotically chi-square with $(\frac{k}{2})$ degrees of freedom. The asymptotic distribution of a related statistic H, proposed by Kruskal and Wallis, is derived under the same assumptions, confirming a result of Kruskal [Ann. Math. Statistics 23, 525–540 (1952); these Rev. 14, 391]. The power of the test is not considered. *W. Hoeffding.*

Lehmann, E. L. The power of rank tests. Ann. Math. Statistics 24, 23–43 (1953).

Nonparametric classes of alternatives are defined for various nonparametric hypotheses in such a way that the

power of any rank test against these alternatives can be rather easily evaluated. For example, let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be independently distributed with cumulative distribution functions (cdf's) F and G respectively. Let H be the hypothesis that $G=F$, where F is an arbitrary continuous cdf, let $g(x)$ be a continuous, nondecreasing function defined over the interval $[0, 1]$ such that $g(0)=0$, $g(1)=1$, and let K be the class of alternatives $G=g(F)$, where F is any continuous cdf. Then the power of any rank test for testing H is constant over K. For the particular choice $g(x)=x^k$ the distribution of the ranks under the alternative is expressed in a simple form, which makes it possible to compare the power of various rank tests against the alternatives under consideration. For another choice of $g(x)$ the Wilcoxon test is found to be most powerful. Similar results are obtained for tests of independence and of other nonparametric hypotheses. *W. Hoeffding.*

Fraser, D. A. S. Nonparametric tolerance regions. Ann. Math. Statistics **24**, 44–55 (1953).

The author continues his investigations in the block technique for constructing confidence and tolerance regions [cf. same Ann. **22**, 372–381 (1951); these Rev. **13**, 260]. Given a space W with an unknown probability distribution, and a sample $w_1, \cdots, w_n$ from this distribution, this sample can be used in many ways for dividing W into blocks, the probability coverages of which have a simple joint distribution. The division may be performed by means of arbitrary functions $\varphi_k(w)$ and integers p_k $(k=1, \cdots, n)$, the kth cut being the surface $\varphi_k(w)=\text{const.}$ passing through the point w_i in which $\varphi_k(w)$ attains its p_kth largest value. The main points in the present paper are (a) that blocks need not be cut off one by one but in groups which are afterwards subdivided, and (b) that the $\varphi_k(w)$ as well as the p_k may depend on the values of the preceding φ_k's. The modifications required when the fundamental probability distribution is discontinuous are indicated. *G. Elfving* (Helsingfors).

Young, Gail S., Jr. A footnote to "Statistical decision functions". Michigan Math. J. **1** (1952), 186–188 (1953).

A somewhat different proof of Theorem 2.22 of A. Wald's "Statistical decision functions" [Wiley, New York, 1950; these Rev. **12**, 193]. *J. Wolfowitz* (Ithaca, N. Y.).

Allen, S. G., Jr. A class of minimax tests for one-sided composite hypotheses. Ann. Math. Statistics **24**, 295–298 (1953).

The one-sided hypotheses referred to in the title concern the parameter θ of a family of densities of the form $p(x, \theta)=w(\theta)\psi(x)\exp \theta x$ from which a sample of a predetermined number of independent observations is drawn. *L. J. Savage* (Chicago, Ill.).

Weiss, Lionel. Testing one simple hypothesis against another. Ann. Math. Statistics **24**, 273–281 (1953).

A "generalized sequential probability ratio test" is defined, which differs from the Wald test only in that the limits at the nth observation are functions of n. Let $f_1(x)$ and $f_2(x)$ be density functions with a finite number of discontinuities. Let T be any test for deciding between $f_1(x)$ and $f_2(x)$ and let $\alpha(T)$ and $\beta(T)$ be the probabilities of the two kinds of error. Let $D_i(n)$, $i=1, 2$, be the distribution function of the number of observations required to make a decision when $f_i(x)$ is actually the distribution. The author proves that there exists a generalized sequential probability ratio test G with probabilities of error $\alpha(G)$ and $\beta(G)$, and distribution function $H_i(n)$, $i=1, 2$, of the number of observations required to make a decision when $f_i(x)$ is actually the distribution, such that $\alpha(G) \leqq \alpha(T), \beta(G) \leqq \beta(T)$, and $H_i(n) \geqq D_i(n)$ for every n and $i=1, 2$. *J. Wolfowitz* (Ithaca, N. Y.).

Schäfer, Wilhelm. Bayes-Funktion ohne Hypothese. Mitteilungsblatt Math. Statist. **5**, 70–74 (1953).

The author discusses and upholds the notions of fiducial distribution and likelihood as originally developed by R. A. Fisher [Proc. Cambridge Philos. Soc. **26**, 528–535 (1930)]. He gives an approximate and an exact fiducial distribution for p of the binomial distribution, and gets the usual maximum likelihood estimate of p from each. *S. W. Nash.*

Kanô, Seigo. On the prediction problem of a stationary stochastic process. Mem. Fac. Sci. Kyūsyū Univ. A. **6**, 173–178 (1952).

The author considers the prediction problem as applied to an individual function f whose (Wiener) spectral distribution exists and has a density. If this density is factorable in a certain way, the (linear least squares) prediction of a weighted average of f taken over a finite segment of time $\geqq t$ in terms of a finite segment of the past before time t is found explicitly. *J. L. Doob* (Urbana, Ill.).

Matschinski, Matthias. Sur les compositions applicables à l'estimation de la probabilité d'une hypothèse. C. R. Acad. Sci. Paris **236**, 1849–1851 (1953).

Mohnsame, Mathias. Processus stochastiques et interprétation géométrique des équations de M. Matschinski. C. R. Acad. Sci. Paris **236**, 1851–1853 (1953).

Sampford, M. R. The estimation of response-time distributions. I. Fundamental concepts and general methods. Biometrics **8**, 13–32 (1952).

The author discusses various situations for which the measurement of response time is of interest, and describes the practical difficulties encountered. The normal case is worked out in detail. *P. Whittle* (Wellington).

Sampford, M. R. The estimation of response-time distributions. II. Multi-stimulus distributions. Biometrics **8**, 307–369 (1952).

The author examines the estimation of the response-time due to a particular stimulus, when other nuisance stimuli are also acting. Two cases are discussed in detail: that in which any response may be related to its appropriate stimulus, and that in which the exact cause of response cannot be determined, but the distribution of "potential response-times" to the stimuli can be assumed normal. Tables are given to ease the calculation of maximum likelihood estimates for these situations. *P. Whittle.*

Torgerson, Warren S. Multidimensional scaling. I. Theory and method. Psychometrika **17**, 401–419 (1952).

The author develops routine procedures for analysing the results of psychophysical experiments such as that described by M. W. Richardson [Psychol. Bull. **35**, 659–660 (1938)]. Stimuli are presented to each subject in threes, and the proportion of subjects judging the kth stimulus to be more similar to the jth than to the ith is denoted by ${}_kp_{ij}$. Let ${}_kx_{ij}$ be such that prob $[x<{}_kx_{ij}]={}_kp_{ij}$, where x is a normal variate with mean 0 and variance 2. (The author's formula (6), defining ${}_kx_{ij}$, should be corrected to agree with his next

sentence.) Presumably the method caters only for experiments yielding no ${}_kp_{ij}$ equal to 0 or 1. Let it be assumed that ${}_kx_{ij}=d_{ik}-d_{kj}+{}_ke_{ij}$, where d_{ik} is a "distance" between the ith and kth stimuli, and the ${}_ke_{ij}$'s are errors. The author shows that the sum of the squares of these is minimized if we take $\frac{1}{2}({}_jx_{.k}+x_{.j}+{}_kx_{.j}+x_{.k})$ as an estimate for $d_{..}-d_{jk}$, where, as usual, a dot indicates averaging with respect to the suffix it replaces. The next step is to estimate $d_{..}$ on the supposition that the d_{jk}'s are equal to distances between pairs of points in a Euclidean space of n dimensions, with n as small as possible. He works out a least-squares estimate of $d_{..}$ for the case $n=1$, and indicates a practicable makeshift for use when n is unknown but the existence of at least one set of four collinear points can be presumed. Lastly, from the estimates for the d_{jk}'s that result he obtains rectangular coordinates for the points (representing stimuli) relative to some set of axes through the centroid of the points: this requires a slight adaptation of the analysis given by G. Young and A. S. Householder [Psychometrika **3**, 19–22 (1938)]. *H. P. Mulholland* (Birmingham).

Whittle, P. Certain nonlinear models of population and epidemic theory. Skand. Aktuarietidskr. **35** (1952), 211–222 (1953).

Stochastic models for population growth, epidemics, etc., are studied. Let $n_i(t)$ be the number of objects of type i at time t, $\Phi(\theta_1, \theta_2, \cdots, t)=E(e^{\theta_1 n_1+\theta_2 n_2+\cdots})$; Φ satisfies an equation of the form $\partial\Phi/\partial t=L(\theta, \partial/\partial\theta, t)\Phi$. If the operator L is independent of t, the author recommends as a practical solution the operator expansion $\Phi=\sum_0^\infty (t^j/j!)L^j\Phi_0$, where $\Phi_0=\Phi(\theta_1, \theta_2, \cdots, 0)$. Methods of systematic calculation are suggested. In a univariate population of initial size n, conditions are given under which $\mu_1(t)=En(t)=T(t)[1+O(n^{-1})]$, where $T(t)$ is the solution of the usual deterministic model corresponding to the process. Expressions for the mean and variance are given with error terms of relative order n^{-2}. The logistic model and an infection model are considered in detail. *T. E. Harris* (Santa Monica, Calif.).

***Wang, Chueh-ming. On the assumptions of variance component analysis and tests of significance.** Essays and papers in memory of late president Fu SSu-Nien, pp. 555–575. The National Taiwan University, Taipei, 1952. (Chinese)

Expository paper.

Ottestad, Per. On the analysis of variance of percentage fractions. Skand. Aktuarietidskr. **35** (1952), 152–159 (1953).

In a one-way classification, let $p_{ij}=m_{ij}/n_{ij}$ be a fraction or percentage for the jth member of the ith class, derived from a classification of count data. The author discusses four different mathematical models which may apply when the variation of the p_{ij} contains components other than the binomial component and points out their implications for an analysis of variance of p_{ij}. In particular, if the levels of the n_{ij} vary from class to class, he recommends working out the regression $(a+bw_i)$ of $s_i^2(p)$, the mean square within the ith class, on the quantity $w_i=\sum_j 1/n_{ij}$. The weight W_i to be attributed to p_{ij} in the analysis of variance is computed from the equation $1/W_i=a+bw_i$. The extension to a two-way classification is indicated. The idea is ingenious, but further comparison with methods previously given in the literature for determining the weights would be illuminating. *W. G. Cochran* (Baltimore, Md.).

Breny, H. Variance and autocorrelation of thickness in random slivers. Appl. Sci. Research A. **3**, 433–450 (1953).

The paper considers, in a thorough manner, stochastic processes associated with the thickness of random slivers of thread. Two systems are considered, the Poisson system and the Bernoulli. The author analyzes the number of fibers crossing any cross-section of the sliver and considers the variations of the thickness of the sliver defined as the sum of the cross-sectional areas of the fibers at any point of it. *L. A. Aroian* (Culver City, Calif.).

Kruskal, William H. On the uniqueness of the line of organic correlation. Biometrics **9**, 47–58 (1953).

Consider a p-dimensional random (vector) variable X having a mean vector μ and a non-singular covariance matrix $\Sigma=\{\sigma_{ij}\}$ and suppose that the covariances σ_{ij} $(i\neq j)$ are all positive or can be made so by replacing some components of X by their negatives. Let λ be the vector with components $\operatorname{sgn}\sigma_{i1}\cdot\sqrt{\sigma_{ii}}$ and l denote the straight line $x=\mu+\lambda t$ $(-\infty<t<\infty)$. This line has been used by several writers to represent (partially) the distribution of X: the author calls it the diagonal line of organic correlation. It is a diagonal of a p-dimensional interval circumscribing a hyperellipsoid of concentration [cf. H. E. Jones, Metron **13**, 21–30 (1937)]. He characterizes it in two ways. (i) It is the unique line, among those based on μ and Σ, that transforms properly under translation, change of scale, and omission of components, and that also indicates the signs of the covariances. (This is similar to a result given by P. A. Samuelson [Econometrica **10**, 80–83 (1942); these Rev. **3**, 173].) (ii) Let each bivariate marginal distribution of two components of X be normal, and consider predictions, of a component X_j when X_i is given, in the form

$$|X_j-\alpha_j-(\nu_j/\nu_i)(X_i-\alpha_i)|\leqq\beta_{ij}\sqrt{\sigma_{jj}},$$

where the β_{ij}'s are chosen constants with $\beta_{ij}=\beta_{ji}$, the parameters α_i and ν_i are the components of vectors α and ν, and (i, j) has been chosen by lot from the $p(p-1)$ possible ordered pairs of unequal suffixes. Then the probability of a correct prediction is maximized if and only if the line $x=\alpha+\nu t$ coincides with l. *H. P. Mulholland.*

Reich, Edgar, and Swerling, Peter. The detection of a sine wave in Gaussian noise. J. Appl. Phys. **24**, 289–296 (1953).

Wiener [Extrapolation, interpolation, and smoothing of stationary time series, Wiley, New York, 1949; these Rev. **11**, 118] made the first systematic statistical approach to the problem of detection in the presence of noise. In his analysis it is assumed that one wishes to reproduce a signal as closely as possible in a certain sense, knowing only signal plus noise. This is the problem of filtering (i.e., reproducing the signal) which is essentially different from the problem of detection as already realized in the work of Van Vleck and Middleton [J. Appl. Phys. **17**, 940–971 (1946)] and of Dwork [Proc. I. R. E. **38**, 771–774 (1950); these Rev. **12**, 191].

The present paper deals with the problem of finding the "optimum" method of detecting a sine wave of known frequency and amplitude in the presence of the so-called Gaussian noise, taking into account the fact that in practice only a finite sample of observed signal is available. The optimum detection method is defined as that which maximizes the probability of recognizing the presence of a sine wave if one has actually appeared; while the probability of

falsely announcing the presence of a sine wave, if none has actually appeared, does not exceed some prechosen value.

It is shown that when the noise has a flat spectrum, all the relevant information is contained in the amplitude and phase of the Fourier transform of the received sample at the frequency of the sine wave. Almost the same result holds in the case where the noise has an exponentially decaying autocorrelation, except that in this case the values of the observed sample at the end point of the sample also play a role. *S. Ikehara* (Tokyo).

***Bubb, Frank W. Linear noise smoothing and predicting filters.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 167–174. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The author considers the smoothing and prediction problem for discrete time series with a filter consisting of a finite number of weight factors. The first part of the paper is devoted to what the author calls a new operational method and which is in essence: (i) the approximation of $f(t)$ on $-M \leqq t \leqq M$ by discrete data $f_n = f(n)$, $-M \leqq n \leqq M$; (ii) a representation of $\{f_n\}$ by $F(x) = \sum_{-M}^{M} f_n x^n$; and finally (iii) an inversion formula which approximates $f(t)$ from $F(x)$ (or $\{f_n\}$) by interpolation. In the second part of the paper the author applies the mean square error criterion to reduce the prediction and filter problem in the usual way to a finite set of linear equations whose unknowns are the filter weight factors [see N. Levinson, J. Math. Physics **25**, 261–278 (1947); these Rev. **8**, 391]. The reviewer was unable to understand the proposed method for solving these equations; in particular, he did not see how (23′) when substituted in (22) could yield (24), and even if (24) were correct it was not apparent why (26) was the required solution. *R. S. Phillips* (Los Angeles, Calif.).

Mathematical Economics

***Dalkey, Norman. Equivalence of information patterns and essentially determinate games.** Contributions to the theory of games, vol. 2, pp. 217–243. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

Using the model of extensive games proposed by the reviewer [Proc. Nat. Acad. Sci. U. S. A. **36**, 570–576 (1950); these Rev. **12**, 515], the author investigates the equivalence of games under variations in the information partition. Two game structures, i.e., games without payoff or chance probabilities assigned, that differ only in personal information are said to be essentially equivalent if there is a one-to-one correspondence between their distinct pure strategies that preserves the expected payoff for all assignments of the payoff functions and chance probabilities. One game structure is an inflation of another if they differ only in personal information and the more precise information in the first structure could have been deduced from the use of pure strategies with the information in the second structure; a game structure in which no further inflation is possible is completely inflated. Two structures are essentially equivalent if and only if they have the same complete inflations. A necessary and sufficient condition that a game structure have an equilibrium point in pure strategies irrespective of the assignment of the payoff functions and chance probabilities is that in the complete inflation each player is completely informed of each previous move of his opponents and knows at least as much as they knew when they made those moves. *H. W. Kuhn* (Los Angeles, Calif.).

Otter, Richard, and Dunne, John. Games with equilibrium points. Proc. Nat. Acad. Sci. U. S. A. **39**, 310–314 (1953).

The authors repeat the formulation of extensive games proposed by the reviewer with the slight modification that chance events may be correlated. Their main result characterizes the structure of games with equilibrium points in pure strategies irrespective of the assignment of the payoff functions [see review above]. *H. W. Kuhn.*

Roy, René. Les élasticités de la demande relative aux biens de consommation et aux groupes de biens. Econometrica **20**, 391–405 (1952).

The paper discusses the way of measuring the elasticities of collective demand for consumers' goods and those of demand for groups of commodities. For the first argument he introduces the notion of "elasticities of distribution" which is defined by λ^k such that $dr^k/r^k = \lambda^k dr/r$ where r^k is the kth consumer's money income and r the collective income. It is pointed out that if $\lambda^k \equiv 1$ the distribution of income is "permanent", the collective demand for the ith good ($Q_i = \sum_k q_i^k$) is homogeneous of degree one in all the prices (p_i) and r, and that θ^k is such that $r^k = \theta^k r$ is constant, so that not only $\sum_i (Q_i p_i/r) EQ_i/Ep_j + Q_j p_j/r = 0$ but also $\sum_j EQ_i/Ep_j + EQ_i/Er = 0$, where E/EX stands for logarithmic differentiation with respect to X. In this special case the number of observations necessary to measure EQ_i/Ep_j and EQ_i/Er is reduced from n^2-1 in the general case to $n(n-1)$.

For the second argument, the author defines the price and quantity indices of a group of commodities A among all goods in such a way that $dp_a/p_a = \sum \gamma dp/p$ and $dq_a/q_a = \sum \gamma' dq/q$ ($\sum \gamma = \sum \gamma' = 1$). For simplicity, a particular case is discussed where there are four goods and they are consolidated to two groups, A and B. Denoting EQ_i/Ep_j, EQ_i/Er and $Q_i p_i/r$ by η_{ij}, ϵ_i and α_i, he shows that

$$\eta_{aa} = \gamma_1'(\eta_{11}+\eta_{12}) + \gamma_2'(\eta_{21}+\eta_{22});$$
$$\eta_{ab} = \gamma_1'(\eta_{13}+\eta_{14}) + \gamma_2'(\eta_{23}+\eta_{24}); \quad \gamma_i = \gamma_i' = \alpha_i/\alpha_a;$$
$$\alpha_a\eta_{aa} + \alpha_b\eta_{ba} + \alpha_a = 0; \quad \alpha_a\eta_{ab} + \alpha_b\eta_{bb} + \alpha_b = 0,$$
$$\eta_{aa} + \eta_{ab} + \epsilon_a = 0; \quad \eta_{ba} + \eta_{bb} + \epsilon_b = 0$$

(if $\lambda^k \equiv 1$); and $\alpha_a\epsilon_a + \alpha_b\epsilon_b = 1$. Finally, the case in which an arrangement of commodities such that $\eta_{ij} = 0$ for $j > i$ is possible is discussed. *S. Ichimura* (Kyoto).

Tobin, James, and Houthakker, H. S. The effects of rationing on demand elasticities. Rev. Econ. Studies **18**, 140–153 (1951).

Relationships between the demand structure under rationing and in a free market are established by the use of Slutsky's relation. The analysis refers to an individual consumer; his free demand for the rationed commodity is assumed to equal the ration. [For an analysis that is free from these limitations, see S. Malmquist, thesis, Uppsala, 1948; these Rev. **10**, 313.] *H. Wold* (Uppsala).

Morishima, Michio. Consumer's behavior and liquidity preference. Econometrica **20**, 223–246 (1952).

Morishima generalizes the well-established theory of consumer's behavior and analyzes the demand for bonds and money as well as real goods. Most of the recent discussions

in the theory of consumer's choice are given illuminating answers in the framework of his theory. A consumer is assumed to maximize his utility function:

$$U[\varphi(x_1, \cdots, x_n), q(H, B, p_1, \cdots, p_n, p_b; \xi)]$$

subject to his budget equation:

$$\sum p_i x_i + H + p_b B = M + (1+p_b)\bar{B} = I.$$

Here x_i stands for the ith real good, p_i its price, M and H the quantities of cash held by the consumer at the beginning and end of the period, $\bar{B}$ and B the number of bonds at the beginning and end of the period, p_b the price of the bonds (the reciprocal of the rate of interest), and q the probability of the future living standard. In order to introduce such an utility function as given above, he assumes the separability postulate of Sono [Kokumin Keizai Zasshi **74**, no. 3, 1–5 (1943); the same postulate discussed by W. Leontief, Econometrica **15**, 361–373 (1947); these Rev. **9**, 604] and assumes that real goods are separable from money, bonds, and prices; namely, u_i/u_j $(i, j=1, 2, \cdots, n)$ are independent of q where $u_i=\partial U/\partial x_i$. Under usual assumption of $d^2U<0$ subject to $dU=0$ and the properties of the separability postulate, it is proved that the equilibrium values of x_i, H, and B (x_i^0, H^0, and B^0) are also those determined by Max (U) subject to $E\equiv I-H^0-p_bB^0=\sum p_ix_i$, and that

$$\partial x_i/\partial p_k = -x_kX_i+\lambda' X_{ki}+(\partial E/\partial p_k)X_i,$$

where $\lambda'=(\partial U/\partial\varphi)^{-1}U_H$, X_i and X_{ik} are V_i/V and V_{ik}/V, V being

$$\begin{vmatrix} 0 & p_j \\ p_i & \varphi^0_{ij} \end{vmatrix},$$

and V_i, V_{ik} the cofactors of p_i and φ_{ik}. And, accordingly, the reinterpretation of the traditional theory is given as a special case of this theory. On the assumption:

$$\partial(U_B/U_H)/\partial_B<0, \quad \partial(U_B/U_H)/\partial H>0,$$

he determines the signs of two respective terms (income effect and substitution effect) of $\partial H/\partial p_k$ and $\partial B/\partial p_k$ according to the signs of $\partial E/\partial p_k$ and $\partial(U_H^0/U_B^0)/\partial p_k$. As for the effect of a proportional change in all prices, Morishima assumes that U is homogeneous of order zero in H, B, p_i and obtains:

$$dx_i = -I(1-\partial H/\partial I-p_b\partial B/\partial I)X_i d\theta,$$
$$dH = H(1-(I/H)\partial H/\partial I)d\theta,$$

and

$$dB = B(1-(I/B)\partial B/\partial I)d\theta$$

where $d\theta=dp_i/p_i$ for any i. Economic interpretations of these equations are given and the monetary effect of O. Lange [Price flexibility and employment, Principia Press, Bloomington, Indiana, 1944] is discussed. Similarly, the effect of change in p_b is discussed. Finally, the homogeneities of utility function and demand functions and their mutual connections are discussed and D. Patinkin's mistake is pointed out [D. Patinkin, Rev. Econ. Studies **18**, 42–61 (1950)]. *S. Ichimura* (Kyoto).

TOPOLOGY

*Kuratowski, Casimir. **Topologie. Vol. II.** 2ème éd. Monografie Matematyczne, Tom XXI. Polskie Towarzystwo Matematyczne, Warszawa, 1952. viii+443 pp. $6.00.

Reprinted, with corrections, from the 1st edition (Warszawa-Wrocław, 1950; these Rev. **12**, 517].

Gomes, A. Pereira. La fonction diamètre et les structures uniforme et topologique. Portugaliae Math. **12**, 73–85 (1953).

A pseudo-diameter is a function d such that: (0) $x\leqq y$ implies $d(x)\leqq d(y)$; (1) given $a\neq 0$ there is $b\neq 0$ such that if x and y intersect, and $d(x), d(y)\leqq b$, then $d(x\cup y)\leqq a$; (2) if z is the union of a family $\{x, y, \cdots\}$ and $d(x\cup y)\leqq a$ for all pairs in that family, then $d(z)\leqq a$. Here the domain of d can be any lattice, and the range in any partially ordered set with least element 0. It is shown that the concepts of uniform structure, metric, and so forth, can be defined in terms of pseudo-diameter. *R. Arens.*

Nagata, Jun-iti. On conditions in order that two uniform spaces are uniformly homeomorphic. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **2**, 81–87 (1952).

Nagata, Jun-iti. On uniform homeomorphism between two uniform spaces. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **3**, 9–13 (1952).

If $L(R)$ is a class of open uniform coverings of a space R which contains a refinement of every uniform covering of R, then $L(R)$ is called a uniform basis of R. Conditions involving lattice properties and local properties of $L(R)$ which determine R to within a uniform homeomorphism are investigated. The second note employs a class of classes of coverings to simplify the conditions and proofs of the first. *L. W. Cohen* (Washington, D. C.).

Nagata, Jun-iti. Topological spaces and lattices. Sûgaku **4**, 65–75 (1952). (Japanese)

The most part of this paper is published in English in the papers reviewed above. *K. Iwasawa.*

Konishi, Isao. On uniform topologies in general spaces. J. Math. Soc. Japan **4**, 166–188 (1952).

Uniform topology in a space $X=[x]$ is studied in terms of a class $\Psi=[\phi]$ of set-valued functions such that $x\in\phi(x)\subset X$. The space is said to have a Ψ-topology and a generalized uniformity Ψ. The class $[\Psi]$ of generalized uniformities for X is partially ordered by writing $\Psi_1<\Psi_2$ if for $\phi_1\in\Psi_1$ and $x\in X$ there is a $\phi_2\in\Psi_2$ such that $\phi_2(x)\subset\phi_1(x)$. An equivalence $\Psi_1\sim\Psi_2$ is defined by $\Psi_1<\Psi_2$ and $\Psi_2<\Psi_1$. If X is given as a neighborhood space, a Ψ-topology in X is called basic if the set $[\phi(x)\,|\,\phi\in\Psi, x\in X]$ is a basis for the neighborhoods of X. The problems considered are the completion and compactification of an X having a generalized uniformity.

In X with a Ψ-topology a filter $\mathfrak{M}$ is called a Ψ-filter if for $\phi\in\Psi$ there is an $x\in X$ such that $\phi(x)\in\mathfrak{M}$; if $\mathfrak{M}$ is maximal it is called a maximal Ψ-filter. If every maximal Ψ-filter has a Ψ-limit point, X is called Ψ-complete. If X is compact, X is Ψ-complete and Ψ-totally bounded, the converse being true if Ψ is basic.

If $\mathfrak{M}_1$, $\mathfrak{M}_2$ are Ψ-filters in X, $\mathfrak{M}_1<\mathfrak{M}_2$ is defined by the condition that $\phi(x)\in\mathfrak{M}_1$ implies $\phi(x)\in\mathfrak{M}_2$ for all $\phi\in\Psi$, $x\in X$. The relation $\mathfrak{M}_1\approx\mathfrak{M}_2$ defined by $\mathfrak{M}_1<\mathfrak{M}_2$ and $\mathfrak{M}_1>\mathfrak{M}_2$ is an equivalence. If $\mathfrak{M}$ is a maximal Ψ-filter, the class of maximal Ψ-filters equivalent to $\mathfrak{M}$ is denoted by $[\mathfrak{M}]$. For a neighborhood space X, $[\mathfrak{M}]\to x$ is defined so that it implies $\mathfrak{M}\to x$, while the converse holds if Ψ is basic. A class $[\mathfrak{M}]$ is non-convergent if $[\mathfrak{M}]\to x$ is false for all x. The space Z whose elements are the points of X and the elements of the class Z_0 of non-convergent $[\mathfrak{M}]$ is a com-

pletion of X. A generalized uniformity Ψ_Z may be defined for Z so that Z is Ψ_Z-complete and the embedding of X in Z has desirable completion properties. However, it is shown that equivalent generalized uniformities may lead to non-equivalent completions.

If X is Ψ-totally bounded, the completion Z is compact but may contain superfluous points even if Ψ is basic. In order to find a smaller compactification of X a special filter $\mathfrak{M}$ is defined by the conditions that it is a maximal Ψ-filter and if $\mathfrak{M}'$ is a maximal filter such that $\mathfrak{M}' < \mathfrak{M}$ then $M' \subset M$. If $\mathfrak{M}$ is a special filter, then $z = [\mathfrak{M}] \in Z_0 \subset Z$ is a special point of Z. The set R of all points of X and special points of Z is a compact subspace of Z (when X is Ψ-totally bounded) and if X is compact, $R = X$. This compactification has uniqueness properties related to results of N. A. Shanin [C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 6–9 (1943); these Rev. **5**, 45] and K. Morita [Proc. Japan Acad. **27**, 130–137 (1951); these Rev. **14**, 69].

In seeking a completion with fewer superfluous points a Cauchy filter is defined as follows. For $M \subset X$ let $S(M, \phi)$ be the union of all $\phi(x)$ intersecting M. If $\mathfrak{M}$ is a filter and $M^* = [M' \mid M' \supset S(M, \phi), M \in \mathfrak{M}, \phi \in U]$ has the property that for $\phi \in \Psi$ there is an $x \in X$ such that $\phi(x) \in M^*$, M^* is called a Cauchy filter. A Cauchy filter is a Ψ-filter. Of two filters giving rise to the same Cauchy filter it is shown that one but not the other may have a limit in X. If X has a Ψ-topology and Y is the union of X and the set Y_0 of non-convergent M^* arising from maximal M, then Y is called a C-extension of X and has interesting completeness and compactness properties. However, the uniform homeomorph of a Cauchy filter need not be a Cauchy filter.

The concluding section treats the extent to which regularity properties of X are carried over to its C-extension and the completeness of this extension. *L. W. Cohen.*

Smirnov, Yu. M. On a problem connected with the metrizability of topological spaces. Ukrain. Mat. Žurnal **3**, 161–163 (1951). (Russian)

Let X be a locally countably compact Hausdorff space which is the union of countably many subspaces A_n, where each A_n is a G_δ in X and in its relative topology has a countable open basis. Then X is metrizable. The same is true if X is regular and the A_n's are all absolute G_δ's. As a corollary, a problem posed by Aleksandrov and Uryson [Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1 **14**, no. 1 (1929); also P. S. Uryson, Works on topology . . . , vol. II, Gostehizdat, Moscow-Leningrad, 1951, pp. 848–963; these Rev. **14**, 122], as follows. Let R be a compact Hausdorff space and let Φ be the closed subset of R consisting of those points which have no metrizable neighborhood. Then, if R is non-metrizable and Φ is metrizable, $R \cap \Phi'$ is non-metrizable. *E. Hewitt* (Seattle, Wash.).

Plans, Antonio. On dimensional approximation in a Kuratowski space. Revista Acad. Ci. Madrid **46**, 303–306 (1952). (Spanish)

Morita, Kiiti. On the dimension of product spaces. Amer. J. Math. **75**, 205–223 (1953).

It is shown that $\dim (X \times Y) \leq \dim X + \dim Y$ if (i) $X \times Y$ is an S-space or (ii) X is fully normal and Y is locally compact and fully normal or (iii) X is countably paracompact normal and Y is locally compact metric. It is shown that equality holds if (i) X is locally compact fully normal with $\dim X \geq 0$ and Y is fully normal with $\dim Y = 1$ or (ii) X is fully normal with $\dim X \geq 0$ and Y is a locally finite polytope with $\dim Y \geq 0$. Earlier results with stronger hypotheses are due to Hemmingsen [Duke Math. J. **13**, 495–504 (1946); these Rev. **8**, 334] and Begle [Bull. Amer. Math. Soc. **55**, 577–579 (1949); these Rev. **10**, 726]. *A. D. Wallace* (New Orleans, La.).

Watson, P. D. On the limits of sequences of sets. Quart. J. Math., Oxford Ser. (2) **4**, 1–3 (1953).

Kuratowski showed [Topologie I, Warszawa-Lwów, 1933, p. 157] that if X is a separable metric space, then the set $(2^X)_L$ of closed subsets of X, with the Hausdorff sequence topology, is an L^*-space, and gave an example to show that if X is not compact, then $(2^X)_L$ need not be a topological space. The author clarifies the problem by showing that (i) if X is not locally compact, then $(2^X)_L$ is not a topological space; (ii) if X is locally compact, then $(2^X)_L$ is a compact metric space. *M. E. Shanks* (Lafayette, Ind.).

Zimmermann, Wolfhart. Stetige Abbildungen topologischer Räume auf endliche abstrakte Komplexe. Math. Z. **56**, 296–311 (1952).

The author approximates topological spaces as inverse limits in the following fashion. A complex K is, by definition, a finite family of finite sets; K is topologized by the left topology of the order $\subset$. A continuous function f on a topological space X onto K corresponds to a unique open cover of X, under the following construction: for each member p of a member of K let $U(p) = \{x : x \in X \text{ and } p \in f(x)\}$; then the family of sets $U(p)$ is an open cover whose nerve is isomorphic to $\bar{K} = K$ together with all non-void subsets of members of K. The mapping scheme S of X is the family of all continuous maps of X onto complexes contained in the family of finite subsets of a fixed countable set. In a natural way, S is an inverse system of spaces, and (a pleasant feature) the projections are unique. (The construction is actually made for a "topological domain", which is a set X together with a lattice U of subsets that covers; but when representation of X, U as an inverse limit is possible, the homology groups of the limit are identical with those of X, T, where T is topology whose base is U.) The principal results are: Each T_0-space X can be embedded topologically in the inverse limit of $L(X)$ of its mapping scheme. This representation is faithful (a homeomorphism onto) if and only if X is the inverse limit of some system of complexes. The class of faithfully representable spaces is closed under inverse limits. The homology and cohomology theories of X are defined naturally in terms of the mapping scheme, and it turns out that the Čech groups for a T_0-space X are identical with those of $L(X)$.

J. L. Kelley (Berkeley, Calif.).

Freudenthal, H. Enden und Primenden. Fund. Math. **39** (1952), 189–210 (1953).

The author devises a compactification R^* appropriate to metric spaces R restrained merely by a property of "tightness" (bundig): R is bundig if the number of subsets of R which are both open and closed is countable. The process is such that if R is semicompact (= peripherally compact = a plenitude of neighborhoods with compact boundaries), then $R^* - R$ is zero-dimensional. On the other hand, if R is an open domain of some euclidean space, there is an appropriate metric for R such that R^* is the Mazurkiewicz compactification by prime-ends [Fund. Math. **33**, 177–228 (1945); these Rev. **8**, 47].

The construction of this compactification is as follows. If O and P are open subsets of R, O is said to be distant from

P, symbolically $O \Lambda P$. If every decreasing sequence of closed subsets connected between O and P has a non-empty intersection. Next, consider on R the space Φ of all continuous functions f such that $f \in \Phi$ implies that $f \geqq 0$, $\inf f = 0$, and for every pair of real numbers: $0 < a < b$, the set on which $f < a$ is distant from the set where $f > b$. Define $f \vdash g$, f is asymptotically bigger than g, to mean that $\lim f(x_n) = 0$ implies $\lim g(x_n) = 0$ for every sequence x_n. Let $\max(f, g)$, resp. $\min(f, g)$, denote the function $h(x) = \max(f(x), g(x))$, resp. $\min(f(x), g(x))$. If $f, g \in \Phi$, then $\min(f, g) \in \Phi$ but $\max(f, g)$ belongs to Φ if and only if the sets where $f < 1/n$ and $g < 1/n$ have a non-empty intersection, for every integer n.

Let F and G denote subsets of Φ. Then $h \dashv F$ shall mean that $h \vdash f$ for all $f \in F$, and $F \vdash G$ shall mean that $h \in \Phi$ and $h \vdash F$ implies $h \vdash G$. It is shown that the relation $\vdash$ between sets is an equivalence relation. A non-empty $F \subset \Phi$ is called bound if for every $f_1, f_2, \cdots, f_n$ belonging to F, $\max(f_1, \cdots, f_n) \in \Phi$, and is called strongly bound if also $\max(f_i, f_j) \in F$. F is maximal if it is bound and if it is necessarily equivalent to every bound G for which $G \vdash F$. Let F be maximal and denote by ΠF the set of all bound G equivalent to F. For bound G, denote by $A(G)$ the set of all ΠF such that $F \vdash G$. Each ΠF becomes a point of a new space R^*, and the collections $A(G)$ are taken as the closed subsets of R^*. It is proved that the space R^* topologized in this way contains a subset homeomorphic to R, and is the desired compactification of R. The most difficult argument is in the proof that the closed subsets of R^* have a countable basis, so that R^* is a compact separable space.

The author gives a characterization of the space R^* in the following terms. A continuous map θ of R into a compactum S is called a Λ-compactification if $S = \overline{\theta(R)}$ and if for every pair of open sets P and Q of S for which $\bar{P} \cap \bar{Q}$ is empty, the set $\theta^{-1}(P)$ in R is distant from $\theta^{-1}(Q)$. Now R^* can be shown to be the "weakest" such compactification in the sense that if S is another Λ-compactification of R, there will exist a continuous map $\varphi(R^*) = S$ which is a homeomorphism of R into itself (taken first as subspace of R^*, then of S). *L. Zippin* (Flushing, N. Y.).

Burgess, C. E. Continua which are the sum of a finite number of indecomposable continua. Proc. Amer. Math. Soc. **4**, 234–239 (1953).

The following theorems are proved on the basis of R. L. Moore's Axiom 0 and the first three parts of Axiom 1: (a) if n is a positive integer and the compact continuum M is indecomposable under index n, then there is only one collection of indecomposable continua whose finished sum is M; (b) if $n > 1$, then in order that the compact continuum M should be indecomposable under index n, it is necessary and sufficient that M should be the finished sum of n indecomposable continua and be irreducible about some n points. *H. M. Gehman* (Buffalo, N. Y.).

Sitnikov, K. Example of a two-dimensional set in three-dimensional Euclidean space allowing arbitrarily small deformations into a one-dimensional polyhedron and a certain new characteristic of the dimension of sets in Euclidean spaces. Doklady Akad. Nauk SSSR (N.S.) **88**, 21–24 (1953). (Russian)

Let K_k ($k = 1, 2, \cdots$) be the 1-skeleton of the cell-complex into which Euclidean 3-space is subdivided by the planes $x = p/k$, $y = q/k$, $z = r/k$ (p, q, r arbitrary integers). Let Q_k be the set obtained from K_k by a translation, so chosen that all the Q_k are disjoint. The example of the title is the complement of the union of the Q_k. The deformation property is clear, since the complement of each K_k admits a "small" deformation onto the 1-skeleton of the dual subdivision. The dimension property follows from a theorem of the author [same Doklady (N.S.) **83**, 31–34 (1952); these Rev. **13**, 860], stating that any two points in the complement of an at most $(n-2)$-dimensional set in n-space can be connected by a continuum, which is impossible in the present case. This example, according to the author, contradicts a statement by Chogoshvili [Compositio Math. **5**, 292–298 (1937)]. Theorem II states that an r-dimensional set in n-space, with r-dimensional closure, cannot be ϵ-deformed for sufficiently small ϵ, into any polyhedron of dimension less than r. Theorem III states: given an r-dimensional set A in n-space and an open set U, then $A \cap U$ admits a deformation within U which, roughly speaking, is constant on the boundary of U, and whose end result lies on a polyhedron of dimension r, and for some U the number r cannot be replaced by $r-1$. *H. Samelson* (Ann Arbor, Mich.).

Miyazaki, Hiroshi. The paracompactness of *CW*-complexes. Tôhoku Math. J. (2) **4**, 309–313 (1952).

The author gives a simple proof that any (not necessarily simplicial) *CW*-complex in the sense of J. H. C. Whitehead [Bull. Amer. Math. Soc. **55**, 213–245 (1949); these Rev. **11**, 48] is paracompact. Only proofs for the simplicial case had hitherto been published. *J. Dugundji.*

van Est, W. T. A generalization of a theorem of Miss Anna Mulliken. Fund. Math. **39** (1952), 179–188 (1953).

The author gives linking theorems for cycles in S^n of the Mulliken type. These are not entirely new. A stronger version of the extended Mullikin theorem has been given by Milgram [Trans. Amer. Math. Soc. **43**, 465–481 (1938)] who also has a corollary (p. 479) analogous to the author's Theorem III. Aside from the difference in the homology theory used, the arguments are parallel, as might be expected. *A. D. Wallace* (New Orleans, La.).

Wada, Hidekazu. Über die Abbildungen vom Komplexe auf den ungeraden dimensionalen reellen projektiven Raum. Tôhoku Math. J. (2) **4**, 231–241 (1952).

In this paper, the author proves a classification theorem stated as follows: If $n > 1$ is an odd integer, then the homotopy classes of the continuous maps $f: K^n \to P^n$ of an n-dimensional complex K^n into the n-dimensional real projective space P^n are in a one-one correspondence with the elements of the direct sum of the 1-dimensional cohomology group $H^1(K^n; Z_2)$ of K^n over the integers mod 2 and the n-dimensional cohomology group $H^n(K^n; Z)$ over the integers; the correspondence is given by the induced homomorphisms.

It is also shown that the theorem stated above is not true if the dimension n is even. Indeed, for each element a of $\pi_n(P^n)$ and for the generator w of $\pi_1(P^n)$, the relation $w \cdot a = (-1)^{n+1} a$ holds. Hence, if n is even, the homotopy classes of the continuous maps of an n-sphere S^n into P^n are in a one-one correspondence with the non-negative integers 0, 1, 2, $\cdots$. In the last section of the paper, Hopf invariants are defined for the continuous maps of a 3-sphere into the real projective plane P^2. These Hopf invariants determine a one-one correspondence between the homotopy classes of these maps and the integers. *S. T. Hu.*

Clark, C. E. Homologies in a normal space and closed subspace. Publ. Math. Debrecen **2**, 237–243 (1952).

In this paper the author extends one of his early works [Duke Math. J. **5**, 62–71 (1939)]. A relation on homology groups is first established for finite polyhedra, then extended to normal spaces. The relation for finite polyhedra is essentially as follows, and it can be easily proved by using the Eilenberg-Steenrod axioms. Let K be a finite polyhedron and let L and C be subpolyhedra of K. Denote $R=\overline{K-L}$ and $B=R\cap L$ and let $\mathfrak{B}$, $\mathfrak{L}$, $\mathfrak{F}$ be the respective kernels of the natural homomorphisms

$$H_p(B, B\cap C)\to H_p(R, R\cap C),\quad H_p(L, L\cap C)\to H_p(K, C),$$
$$H_p(B, B\cap C)\to H_p(R, R\cap C)\times H_p(L, L\cap C).$$

Then there is a natural isomorphism of $\mathfrak{B}/\mathfrak{F}$ onto $\mathfrak{L}$.

C. T. Yang (Champaign, Ill.).

Olum, Paul. On mappings into spaces in which certain homotopy groups vanish. Ann. of Math. (2) **57**, 561–574 (1953).

Let P be an n-dimensional connected locally-finite polyhedron and let Y be an arcwise-connected topological space whose homotopy groups vanish in dimensions $2, 3, \cdots, n-1$. This paper solves the classification problem for homotopy classes of maps $P\to Y$. The same problem has been solved by M. M. Postnikov [Doklady Akad. Nauk SSSR (N.S.) **67**, 427–430 (1949); these Rev. **11**, 451] in a paper not containing proofs.

A map f: P, $x_0\to Y$, y_0 induces a homomorphism θ: $\pi_1(P, x_0)\to\pi_1(Y, y_0)$, and, conversely, any homomorphism is induced by a map. Moreover, a homotopy class of maps (the base point being allowed to wander during the homotopy) determines a homomorphism class Θ, with respect to inner automorphisms of $\pi_1(Y, y_0)$, and any two maps P, $x_0\to Y$, y_0 giving rise to the same Θ are homotopic on P^{n-1}. Let $\theta\in\Theta$ be chosen arbitrarily. Let K be a fixed triangulation of P; for each vertex a of K, choose a fixed path x_0a. Let $\theta^*\pi_n$ be the local system of groups on K, isomorphic to $\pi_n(Y)$, such that to each ordered 1-cell a_0a_1 is associated the automorphism $\alpha\to\theta(\omega)(\alpha)$, where $\alpha\in\pi_n(Y)$, and ω is the element of $\pi_1(P, x_0)$ represented by the loop $x_0a_0a_1x_0$. Then, using the Eilenberg-MacLane cohomology class $k^{n+1}\in H^{n+1}(\pi_1(Y), \pi_n(Y))$, the author describes a crossed homomorphism β_θ of π_θ, the centralizer of $\theta\pi_1(P, x_0)$ in $\pi_1(Y, y_0)$, into $H^n(K, \theta^*\pi_n)$. This crossed homomorphism β_θ induces a partitioning of $H^n(K, \theta^*\pi_n)$ into cosets, two elements h, $h'\in H^n(K, \theta^*\pi_n)$ going into the same coset if there exists $\eta\in\pi_\theta$ such that $h'=\beta_\theta(\eta)+\eta\cdot h$. The classification theorem asserts that the homotopy classes of maps $P\to Y$ may be placed in 1-1 correspondence with the pairs (C, Θ), where C is a coset in the above sense. The author shows how the non-invariant features of the definition of β_θ may be circumvented.

The problem here discussed may be described as the free homotopy problem, to distinguish it from the problem of classifying homotopies relative to a non-empty subpolyhedron of P. This latter problem was solved by the author [Ann. of Math. (2) **52**, 1–50 (1950); these Rev. **12**, 120]; many of the ideas and results, and the notation of that paper have been used in the present work.

P. J. Hilton (Cambridge, England).

GEOMETRY

Toscano, L. Sur les triangles podaires orthogonaux. Bull. Soc. Roy. Sci. Liège **21**, 550–556 (1952).

Toscano, L. Points remarquables d'un triangle sur le cercle de Brocard et sur la droite de Lemoine. I, II. Bull. Soc. Roy. Sci. Liège **21**, 557–573 (1952); **22**, 47–65 (1953).

Court, N. A. The semi-orthocentric tetrahedron. Amer. Math. Monthly **60**, 306–310 (1953).

Chisini, O. La non dimostrabilità del Postulato di Euclide. (Trattazione elementare.) Period. Mat. (4) **31**, 7–33 (1953).

Lorent, H. Courbes construites à partir de deux lignes dépendant de deux paramètres liés. Bull. Soc. Roy. Sci. Liège **21**, 540–549 (1952).

Cesarec, Rudolf. On circular rational curves of the 4th order derived from certain conoids. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **276**, 39–82 (1949). (Serbo-Croatian)

Cesarec, Rudolf. Sur les quartiques planes unicursales circulaires, dérivées des certains conoïdes. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **5**, 9–23 (1952).

An abbreviated version of the paper listed above.

Jensen, Henry. A short geometric proof. Mat. Tidsskr. A. **1952**, 77–78 (1952). (Danish)

An elegant proof for the fact that the surface generated by revolving one line about a line skew to it contains a second family of lines. *H. Busemann.*

Ylitch-Daïovitch, Militsa. Quelques théorèmes de la géométrie projective. Bull. Soc. Math. Phys. Serbie **4**, no. 3–4, 43–48 (1952). (Serbo-Croatian. French summary)

Bagchi, Hari das, and Chaki, Manindra Chandra. Note on certain remarkable types of plane collineations. Ann. Scuola Norm. Super. Pisa (3) **6**, 85–97 (1952).

If a plane set of points $A_1, \cdots, A_n$ are permuted cyclically by a projectivity of the plane, it is obvious that the n sets of $n-1$ concurrent lines such as those joining A_1 to $A_2, \cdots, A_n$ are all homographically related. It is shown for $n=5, 6$ that this relation between a consecutive three of these n sets is a sufficient condition for $A_1, \cdots, A_n$ to be permuted cyclically by a projectivity. *P. Du Val.*

Cuesta, N. Complex involutory correlation. Revista Mat. Hisp.-Amer. (4) **12**, 330–339 (1952). (Spanish)

Let F' denote the incidence lattice of the set F of linear subspaces of the n-dimensional vector space over the complex field C. Consider transformations induced by $y^*=(a_{ij})(x\phi)^*$ where (a_{ij}) is a non-singular square matrix, x a vector, * is matrix transposition, and ϕ is an automorphism of C. Various detailed results are given, mostly in terms of solutions h of $(a_{ij}\phi^{-1})^*=h(a_{ij})$. Theorem: If n is

odd, the involutory correlations of F' are all polarities ($h=1$). If n is even, the involutory correlations of F' resulting from a ϕ other than the identity are polarities; of those resulting from $\phi=I$ some are polar, others are focal ($h=-1$). *P. M. Whitman* (Silver Spring, Md.).

Lagrange, René. Quelques problèmes sur les produits d'inversions. Ann. Sci. Ecole Norm. Sup. (3) **69**, 83–108 (1952).

Let $U, X, \cdots$ denote $(n-2)$-spheres in $(n-1)$-space and let $\bar{U}$ denote the inversion with respect to U. Define $\overline{U_m\cdots U_2U_1}$ by $\overline{U_m\cdots U_2U_1}X=\bar{U}_m(\cdots(\bar{U}_2(\bar{U}_1X))\cdots)$. Suppose $U_1, \cdots, U_n$ are linearly independent and the n-dimensional pencil α of $(n-2)$-spheres spanned by them contains an $(n-2)$-dimensional subpencil β such that each $(n-2)$-sphere of β is invariant with respect to $\overline{U_n\cdots U_2U_1}$. Then β contains $n-2$ mutually orthogonal $(n-2)$-spheres $X_1, \cdots, X_{n-2}$ and α contains two $(n-2)$-spheres X_{n-1} and X_n perpendicular to β such that

$$\overline{U_n\cdots U_2U_1}=\overline{X_nX_{n-1}X_{n-2}\cdots X_2X_1}.$$

The pencil β consists of the $(n-2)$-spheres through the circle $D=X_1\cap X_2\cap\cdots\cap X_{n-2}$. The transformation $\overline{X_{n-2}\cdots X_2X_1}$ is an "anallagmatic reflection" with respect to D, while $\overline{X_nX_{n-1}}$ is interpreted as an "anallagmatic rotation" about the $(n-3)$-sphere $X_n\cap X_{n-1}$. Special cases and related special problems are considered in detail. *P. Scherk.*

Hanani, Haim. On a point of minimum sum of distances from given points. Riveon Lematematika **6**, 24–25 (1953). (Hebrew. English summary)

The author settles the simple case in which the point in question does not coincide with one of the given points.

E. G. Straus (Los Angeles, Calif.).

Convex Domains, Extremal Problems, Integral Geometry

Steinberger, Mëir. On the diameter of a polygon and on the diameters of the polygons into which it may be divided. Riveon Lematematika **6**, 34–40 (1953). (Hebrew. English summary)

The author discusses conditions under which the diameter $\delta(V)$ of a (not necessarily connected) polygon V is less than (or is equal to) the sums of the diameters of subpolygons which compose V. He applies the results to the concept of "degree of equivalence" $\sigma(V, W)$ of polygons V, W, i.e., the least number of pairwise congruent polygons into which V and W can be decomposed. A consequence of his theorem is: If V is connected, then $\sigma(V, W)\geqq\delta(V)/\delta(W)$. If V is convex and incongruent to W, then the strict inequality holds. The author claims that this improves on results by A. Tarski [Młody Matematyk **1**, 37–44 (1931); Riveon Lematematika **5**, 32–38 (1951); these Rev. **13**, 577].

E. G. Straus (Los Angeles, Calif.).

Eggleston, H. G. Some properties of triangles as extremal convex curves. J. London Math. Soc. **28**, 32–36 (1953).

The following results are proved concerning the diameters and sixpartite points of a plane convex body K with area a. The area swept out in a half rotation of an essential diameter of K is $\leqq 3a$, equality holding if and only if K is triangular. K is triangular if and only if exactly three diameters pass through each interior point. Every sixpartite point of K is the center of an affinely regular hexagon in K whose area is $\frac{2}{3}a$ and whose sides are bisected by the sixpartition lines. Any line through a sixpartite point of K divides K into two parts with areal ratio at least 4:5; if there is a line for which this ratio is attained, K is triangular. A triangle has only one sixpartite point, its median point. *W. Gustin.*

Eggleston, H. G. On triangles circumscribing plane convex sets. J. London Math. Soc. **28**, 36–46 (1953).

The following is a partial list of inequalities obtained in this paper involving the length l, width w, and area a of a plane convex body K, denoted by K_c if centrally symmetric, and a circumscribing "triangle" (with two sides possibly parallel) Δ, denoted by Δ_e if equilateral: $w\Delta\leqq lK/3^{1/2}$, $w\Delta\leqq lK_c/2$, $a\Delta\leqq aK\cdot 2$ (for some Δ), $l\Delta_c\leqq lK\cdot 3^{1/2}/\pi$ (for some Δ_c), $l\Delta_c\leqq lK\cdot 2$, $l\Delta_c\leqq lK_c\cdot 3^{1/2}$, $l\Delta_e\geqq lK_c\cdot 3/2$.

W. Gustin (Bloomington, Ind.).

Radziszewski, Constantin. Sur un problème extrémal relatif aux figures inscrites dans les figures convexes. C. R. Acad. Sci. Paris **235**, 771–773 (1952).

It is shown that in any plane convex body K a rectangle R can be inscribed having at least half the area of K. By area-preserving parallel-pushing K is first flattened against a line x, then against an orthogonal line y, so as to form a convex body k_{xy} filling an xy quadrant corner. For k_{xy} a desired inscribed rectangle r_{xy} obviously exists, which arises from a certain not necessarily unique parallelogram R_{xy} inscribed in K. By a semicontinuity argument $R_{xy}=R$ is rectangular for some xy position. *W. Gustin.*

Fejes Tóth, L. Ein Beweisansatz für die isoperimetrische Eigenschaft des Ikosaeders. Acta Math. Acad. Sci. Hungar. **3**, 155–163 (1952). (Russian summary)

In 1841, Steiner [Ges. Werke, Bd. 2, Reimer, Berlin, 1882, pp. 177–308] proved that the regular octahedron has a smaller surface for its volume than any topologically isomorphic polyhedron, and conjectured the same result for the dodecahedron and the icosahedron. The case of the dodecahedron was proved by M. Goldberg [Tôhoku Math. J. **40**, 226–236 (1935)] and the author [Amer. J. Math. **70**, 174–180 (1948); these Rev. **9**, 460], but the case of the icosahedron remains open. The author proved [Canadian J. Math. **2**, 22–31 (1950); these Rev. **11**, 386] that, if a polyhedron with an inscribed sphere has e vertices, k edges, f faces, surface F, and volume V, then

$$\frac{F^3}{V^2}\geqq 9k\sin\frac{f\pi}{k}\left(\tan^2\frac{f\pi}{2k}\tan^2\frac{e\pi}{2k}-1\right),$$

with equality if and only if the polyhedron is regular. In particular, F^3/V^2 is smaller for the regular icosahedron than for any other polyhedron with $e=12$, $k=30$, $f=20$, whose faces all touch a sphere. If the stipulation about touching a sphere could be removed, Steiner's conjecture would be proved. It is known [L. Lindelöf, Bull. Acad. Imp. Sci. St. Pétersbourg **14**, 257–269 (1869)] that the "best" polyhedron with a given number of faces has an inscribed sphere; but as the "best" icosahedron is not isomorphic to the regular icosahedron, this does not help. As a background for another approach, the author gives a new and elegant proof that the "best" n-gon in the plane is regular. The analogous procedure in space does not entirely dispose of the problem, but reduces it to an apparently simple minimal property of the regular pentagonal pyramid.

H. S. M. Coxeter (Toronto, Ont.).

Bordoni, Piero Giorgio. Nocciolo di sicurezza di una figura piana convessa. Ricerca Sci. **23**, 69–74 (1953).
Bordoni, Piero Giorgio. Nocciolo di sicurezza di una figura piana convessa. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 242–245 (1952).

If all the vectors going from the center of gravity of a convex homogeneous plane region to its boundary are reduced in the ratio 1/6, it can be proved that the locus of the ends of the reduced vectors is a curve C which has no points external to the nucleus of inertia. The region inside C is called the safety nucleus. It is possible in this way to give a sufficient condition in order that the stresses on the normal section of a cylinder under eccentric compression be of the same sign. *O. Bottema* (Delft).

Krafft, Maximilian. Ein neuer Beweis des Vierscheitelsatzes. Arch. Math. **4**, 43–44 (1953).

A simple elementary proof of the four-vertex theorem based on a lemma on convex arcs of angular measure π, which are tangent at the endpoints. Although no reference is given, the lemma is a special case of a theorem of Süss given by Vietoris [Arch. Math. **3**, 304–306 (1952); these Rev. **14**, 577] and the proof is identical with that of Vietoris. *S. B. Jackson* (College Park, Md.).

Algebraic Geometry

Morgantini, Edmondo. Sulla teoria diametrale e sulla ricerca dei centri e degli assi di simmetria delle curve algebriche piane. Ann. Univ. Ferrara. Sez. VII. (N.S.) **1**, 55–79 (1952).

The theory of polars of a plane curve, and that of polar envelopes dual to it, is called in to obtain joint covariants of a given curve, the line at infinity and the isotropic point pair at infinity; hence, metrical covariants. These are in turn applied to restrict the types of affine or orthogonal transformations which can leave a given curve invariant. The majority of the results are obvious from quite simple metrical considerations, and the statement being somewhat scrappy it is not easy to pick out the few that may be new. *P. Du Val* (Bristol).

Gambier, Bertrand. Sur une propriété projective d'un couple de coniques. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 337–341 (1952).

A conic Σ and four points A, B, C, D being given in a plane, it is known that if A, B, C, D lie on a conic S which contains an intersection of tangents to Σ from A and B, and an intersection of tangents to Σ from C and D, then A, B, C, D lie on a conic (in general unique) which has double contact with Σ. It is now shown that if S is quadrilaterally circumscribed to Σ, then there exist in general two distinct conics through A, B, C, D which have double contact with Σ. *J. G. Semple* (London).

Godeaux, Lucien. Note sur quelques éléments associés aux points d'une surface. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 14–23 (1953).

En utilisant la suite de Laplace associée dans l'espace S_5 à une surface rapportée à ses asymptotiques, on attache à un point de cette surface six quadriques et deux couples de droites. *Author's summary.*

Godeaux, Lucien. Note sur l'enveloppe des quadriques de Lie d'une surface. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 156–164 (1953).

Etude de la configuration formée par certaines suites de Laplace de l'espace à cinq dimensions attachées à une surface de l'espace ordinaire lorsqu'il y a conservation des asymptotiques sur les nappes de l'enveloppe des quadriques de Lie de cette surface. *Résumé de l'auteur.*

Godeaux, Lucien. Sur les surfaces ayant mêmes quadrilatères de Demoulin. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 245–254 (1953).

Recherche des couples de surfaces ayant même quadrilatère de Demoulin, c'est-à-dire des surfaces dont les quadriques de Lie en deux points homologues se touchent en quatre points, caractéristiques pour ces deux quadriques. Étude d'un cas particulier. *Résumé de l'auteur.*

Godeaux, Lucien. Sur un faisceau de surfaces desmiques généralisées. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 232–244 (1953).

Étude d'un faisceau de surfaces d'ordre $4n$ contenant trois surfaces dégénérées. Formation du système adjoint à ce faisceau dont les surfaces, pour $n>1$, sont irrégulières. *Résumé de l'auteur.*

Godeaux, Lucien. Sur quelques points de diramation de seconde espèce et de troisième catégorie d'une surface multiple. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 898–907 (1952).

[For part I see same Bull. (5) **38**, 755–765 (1952); these Rev. **14**, 579.] The example considered in this paper is of a cyclic involution of order 101 on a surface F, with a united point 0 at which (in the author's usual notation) [cf. Acad. Roy. Belgique. Cl. Sci. Mem. Coll. in 8° (2) **27**, no. 1626 (1952); these Rev. **14**, 578] $\alpha=79$, $\beta=78$ and the lowest value of $\lambda+\mu$ is given by $\lambda_1=9$, $\mu_1=5$, so that the point is of the third category and we expect the corresponding point $0'$ on a model Φ of the involution to have four constituents. In fact it is found that the neighbourhood of $0'$ can be analysed into curves σ_1, ρ_1, ρ_2, τ_1, τ_2, ω_1, ω_2, σ_2 intersecting consecutively, and all of grade -2 except τ_1, τ_2, and these of grade -4, -3 respectively. Thus the tangent cone consists of three planes corresponding to σ_1, τ_2, σ_2, and a quadric cone corresponding to τ_1. There are ordinary binodes in the first neighbourhood, at the intersection of the first plane and the quadric, represented by ρ_1, ρ_2 and at the intersection of the other two planes, represented by ω_1, ω_2. *P. Du Val* (Bristol).

***Godeaux, L. Les singularités des points de diramation isolés des surfaces multiples.** Deuxième Colloque de Géométrie Algébrique, Liège, 1952, pp. 225–241. Georges Thone, Liège; Masson & Cie, Paris, 1952. 375 Belgian francs; 2625 French francs.

This paper is a revised and (the author claims) simplified proof of the central theorem of an earlier paper [Acad. Roy. Belgique. Cl. Sci. Mem. Coll. in 8° (2) **27**, no. 1626 (1952); these Rev. **14**, 578], namely, that at an isolated branch point of a cyclic multiple surface of the second species and first, second, or third category, the tangent cone has respectively two, three, or four constituents. *P. Du Val* (Bristol).

Godeaux, Lucien. Observations sur les points unis de seconde espèce et de troisième catégorie des involutions cycliques appartenant à une surface algébrique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 1118–1124 (1952).

Godeaux, Lucien. Les transformations birationnelles du plan. Mémor. Sci. Math., no. **122**. 2nd ed. Gauthier-Villars, Paris, 1953. 70 pp. 900 francs.

This monograph appears to be intended as a fairly easy and elementary introduction to the classical theory of plane Cremona transformations. The first brief chapter contains a résumé of Enriques' treatment of superlinear plane curve branches by their Puiseux expansions, the second an equally brief summary of the general properties of linear systems of plane curves, leading up to the characters of a homaloidal net. Chapter III contains a simple treatment of the general plane Cremona transformation, and the numerical relations between the characters of the homaloidal net and the exceptional curves fundamental to it are obtained rather lucidly by considering the projective model of the sum of the homaloidal net of the transformations with that of lines—a surface which has the merits of being symmetrically related to the given transformation and its inverse and of containing all the fundamental curves of both explicitly as curves. The possibility of some base points being in the neighbourhoods of others, and of some fundamental curves being reducible, is touched on. Chapter IV is a careful and fairly clear study of the Noether-Chisini theorem. The remaining two chapters are devoted to the elementary properties of Cremona-covariance of such things as the Jacobian and adjoint systems of a given one, to the search for minimum-order models of systems of given genus, etc., and to finite continuous groups of Cremona transformations. In these connexions the classical results are stated without much proof, but with adequate references. The book ends with what seems to be a very copious bibliography.

P. Du Val (Bristol).

Knight, A. J. On a class of surfaces with cyclic intersection groups. J. London Math. Soc. **28**, 210–214 (1953).

An example is given to disprove a conjecture of D. B. Scott [Proc. London Math. Soc. (2) **51**, 308–324 (1950); these Rev. **12**, 356] that, in the notation of Scott's paper, Ω is a multiple of a canonical curve in the case $t=1$. The example is the surface mapping the unordered point pairs of a curve of general modulus. *R. J. Walker.*

Burniat, Pol. Sur les surfaces canoniques quadruples (note complémentaire). Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 19–23 (1952).

L'A. observe que dans un récent travail [Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **24**, no. 1602 (1950); ces Rev. **12**, 438] il a partiellement basé une conclusion essentielle sur une propriété incorrectement formulée. Dans cette note, l'A. précise l'énoncé de cette propriété (no. 3) et corrige sur cette base rectifiée la démonstration de la conclusion en question. *M. Piazzolla Beloch* (Ferrara).

***Burniat, Pol. Modèles de surfaces canoniques normales de S_3 et de genre linéaire $11 \leq p^{(1)} \leq 17$.** Deuxième Colloque de Géométrie Algébrique, Liège, 1952, pp. 185–210. Georges Thone, Liège; Masson & Cie, Paris, 1952. 375 Belgian francs; 2625 French francs.

Ce mémoire est un exposé d'ensemble des connaissances actuelles de l'auteur sur les surfaces canoniques de genre $p_g=4$. On y retrouve, notablement simplifiées et améliorées sur certains points, complétées et généralisées sur d'autres, toutes les considérations développées par l'auteur dans Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 880–896 (1950); **37**, 241–251, 367–377 (1951) [ces Rev. **13**, 63, 769].

Les paragraphes A et B sont consacrés à l'étude de la façon dont une surface F^n normale de genre $p_g=0$ permet, en la munissant d'éléments de diramation convenables, de former une surface double $2F^n$ canonique, qui est le représentant dégénéré de tout une classe de surfaces canoniques F^{2n}.

Le paragraphe C contient les résultats: 1) Aux chapitres 11 et 12, on retrouve les deux familles de surfaces F^{10} de genre $p^{(1)}=11$ déjà obtenues dans le troisième mémoire cité. Mais l'amélioration technique de sa méthode permet à l'auteur de présenter en outre trois nouvelles familles de surfaces F^{10} canoniques, admettant comme les précédentes un trièdre de droites quadruples et une biquadratique double, dont les genres linéaires respectifs sont $p^{(1)}=12$, $p^{(1)}=13$, et $p^{(1)}=14$.

2) Au chapitre 13, on retrouve les surfaces canoniques F^{12} de genre $p^{(1)}=13$ qui constituaient la première famille du premier mémoire cité. Les chapitres 14 à 17 sont consacrés à la présentation de trois familles nouvelles de surfaces canoniques F^{12} admettant une droite sextuple D et trois droites quadruples concourantes avec D et coplanaires. Elles ont les genres linéaires respectifs: $p^{(1)}=13$, $p^{(1)}=14$ et $p^{(1)}=15$.

3) Les chapitres 18 et 19 sont consacrés à deux familles de surfaces canoniques F^{14}. L'une d'elles, de genre $p^{(1)}=15$, est la seconde famille du premier mémoire cité. L'autre, qui en diffère par la structure de ses éléments doubles, est nouvelle. Son genre linéaire est $p^{(1)}=16$.

4) Aux chapitres 20 à 22, on retrouve les surfaces canoniques F^{16} de genre $p^{(1)}=17$ qui constituaient la troisième famille dans le second mémoire cité.

L'exposé se termine en démontrant que toutes les surfaces obtenues sont régulières, et en citant celles qui donnent lieu à la critique de Zariski. *L. Gauthier* (Nancy).

Du Val, Patrick. Regular surfaces of genus two. I. Canadian J. Math. **5**, 216–237 (1953).

Dans ce mémoire l'auteur s'occupe des surfaces algébriques régulières ayant le genre $p=p_g=p_a=2$ et le genre linéaire $p^{(1)}=n+1>1$; c'est un type de surfaces qui n'est pas encore complètement connu dans tous ses détails. On sait que la surface considérée possède un modèle bicanonique bien déterminé; c'est à ce modèle que l'auteur adresse son attention, dans les cas $n=1, 2, 3$. Si l'on suppose que le système canonique de la surface considérée soit irréductible, son modèle bicanonique est une surface F^{4n} de l'espace à $n+2$ dimensions; il y a un cône Γ^2_{n+1}, dont le sommet a la dimension $n-1$, qui passe par F^{4n}; les courbes canoniques de F^{4n} sont de l'ordre $2n$ et sont contenues dans les espaces générateurs du cône Γ^2_{n+1}; elles possèdent n points-de-base $A_1, A_2, \cdots, A_n$ placés sur le sommet du cône. Dans certains cas, la surface F^{4n} est l'intersection du cône Γ^2_{n+1} avec une variété G^2_2, touchant Γ^2_{n+1} le long d'une courbe rationnelle normale K^n contenue dans un des espace générateurs du cône, et dont les courbes sections par des espaces S_n sont des courbes canoniques de genre $n+1$. Dans d'autres cas la surface F^{4n} est la base d'un système linéaire de quadriques ayant la dimension $\frac{1}{2}(n-1)(n-2)$. L'auteur expose encore quelques autres propriétés générales qui se rapportent aux points $A_1, A_2, \cdots, A_n$ et aux projections de F^{4n} sur des espaces de dimension plus petite que $n+2$. Après tout cela,

il aborde le problème de la recherche des types différents de surfaces F^{4n}, problème qui se complique rapidement lorsque n croît. Les surfaces F^{4n} sont toutes connues pour $n=1$ et $n=2$; l'auteur en répète ici la description, que l'on peut trouver déjà, par exemple, dans le traité de F. Enriques, Le superficie algebriche [Zanichelli, Bologna, 1949; ces Rev. 11, 202]; il remarque encore que, au contraire de ce que l'on peut lire à p. 315 du traité de F. Enriques, les deux surfaces F^{4n} que l'on a pour $n=2$ sont l'une un cas particulier de l'autre. La discussion du cas $n=3$ conduit à quatre types différents de surfaces; le premier est donné pour une surface F^{12} de l'espace à 5 dimensions, qui est l'intersection d'un cône Γ_4^3 et de deux hypersurfaces cubiques convenablement choisies; les autres types sont, à quelques points de vue, plus particuliers, sans que l'on puisse affirmer qu'il dérivent du premier type par des passages à la limite.

E. G. Togliatti (Gênes).

Brusotti, Luigi. La "piccola variazione" nei suoi aspetti e nel suo ufficio. Boll. Un. Mat. Ital. (3) **7**, 430–444 (1952).

Expository lecture which traces the method of small variations in the theory of real algebraic varieties back to Newton. Most modern work in this field is Italian. Ample references, and some examples.

D. Pedoe (Khartoum).

Baldassarri, M. Le $I^3_{n,2}$ ed una classe di varietà rappresentative. Ann. Univ. Ferrara. Sez. VII. (N.S.) **1**, 143–151 (1952).

Ce mémoire fait suite à deux publications du même auteur [Rend. Sem. Mat. Univ. Padova **21**, 124–135 (1952); Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 530–536 (1952); ces Rev. **14**, 313, 790].

Une involution $I^3_{n,2}$ du plan étant donnée, les résiduels d'un point P décrivent une courbe $\gamma(P)$. Le cas où $\gamma(P)$ dépend d'un seul paramètre a été examiné dans le premier mémoire cité. L'auteur étudie ici le cas où $\gamma(P)$ dépend de deux paramètres. Les résiduels étant représentés sur la variété de Bordiga image des G_{n-1} du plan, $\gamma(P)$ a pour image une courbe rationnelle ψ, qui décrit, lorsque P varie, un système d'indice un dans une V_3 sur laquelle $I^3_{n,2}$ est représentée par une involution J_n^3. Cette représentation de $I^3_{n,2}$ par J_n^3 est birationnelle si les ψ ne sont pas pluri-sécantes à leurs transformées par J_n^3.

La définition d'une $I^d_{n,h}$ normale dans S_H $(h \leqq H \leqq d)$ a été donnée dans le second mémoire cité: l'auteur montre ici, comme application du théorème précédent, que pour les $I^3_{3,2}$ les deux valeurs de H peuvent se présenter. Si $\gamma(P)$ est elliptique, ou hyperelliptique de genre pair, $H=2$; si $\gamma(P)$ est rationnelle, ou hyperelliptique de genre impair, $H=3$.

L'auteur étudie ensuite la représentation des $I^3_{2,2}$ par une sous-variété V_3 de la variété M_4^3 de S_5 lieu des cordes d'une surface de Veronese F_2^4. Les courbes ψ images des $\gamma(P)$ sont découpées sur V_3 par les plans tangents à F_2^4. Il parvient alors à démontrer que toute $I^3_{2,2}$ admet un modèle $I^3_{2,3}$ birationnellement équivalent. Il en résulte notamment qu'une variété V_3 qui admet un automorphisme birationnel involutif J_2^3, et qui contient une congruence d'indice un de courbes rationnelles non invariantes par J_2^3 est unirationnelle, représentable par une $I^3_{2,3}$.

L. Gauthier (Nancy).

Conforto, Fabio. Problèmes résolus et non résolus de la théorie des fonctions abéliennes dans ses rapports avec la géométrie algébrique. Deuxième Colloque de Géométrie Algébrique, Liège, 1952, pp. 89–110. Georges Thone, Liège; Masson & Cie, Paris, 1952. 375 Belgian francs; 2625 French francs.

Si espone in linee essenziali la costruzione delle teorie delle funzioni abeliane definite come funzioni meromorfe di p variabili complesse dotate di $2p$ periodi indipendenti, secondo il programma già sommariamente indicato da Lefschetz e completamente realizzato dall'A. [Funzioni abeliane e matrici di Riemann. Parte Prima. Liberia dell' Università di Roma, 1942; questi Rev. **10**, 29]. Osservato come il metodo proposto permetta di giungere alla caratterizzazione delle matrici di Riemann (matrici di periodi primitivi di un corpo K di funzioni abeliane) ed alla costruzione delle funzioni abeliane come quozienti di serie θ, introdotte così in modo affatto naturale, l'A. si sofferma sui legami tra la teoria delle funzioni abeliane e la geometria algebrica, accennando alle varietà abeliane (rappresentate parametricamente mediante funzioni di un corpo K di funzioni abeliane) ed in particolare alla V_p di Picard del corpo K, e ad alcuni problemi ancora aperti relativi a tali varietà. Ancora insoluto è il problema della caratterizzazione topologica delle V_p; dal punto di vista della geometria algebrica è nota la caratterizzazione, dovuta a Severi, delle varietà di Picard come varietà p-dimensionali che posseggano un gruppo continuo ∞^p di trasformazioni birazionali in sé, ma sarebbe di grande interesse dimostrare se sia anche vera la presunzione di Severi (verificata per $p=1$, 2) secondo cui le V_p di Picard riescano caratterizzate dal possedere serie canonica d'ogni dimensione d'ordine zero. Altre questioni che si pongono sono quella di classificare le involuzioni tracciate sulla V_p di Picard (con $p \geqq 3$) ed inversamente di determinare le varietà algebriche che posseggano una involuzione birazionalmente equivalente ad una varietà di Picard; ed il problema di determinare tutte le sottovarietà della V_p, problema strettamente legato con quello di determinare tutte le varietà algebriche per cui esista un modello in V_p.

Successivamente l'A. accenna al primo problema fondamentale della teoria delle funzioni abeliane modulari, già risolto dal Siegel in un caso particolare; esso consiste nella costruzione di un sistema completo di rappresentanti delle classi di equivalenza delle matrici ω di Riemann, rispetto alla relazione $\omega' = \alpha\omega A$ (α matrice non degenere, A matrice intera unimodulare), o, ciò che é lo stesso, di un sistema completo di rappresentanti per le classi di varietà di Picard birazionalmente distinte. Alla risoluzione di tale problema è dedicato un recente volume dell'A. [F. Conforto, Funzioni abeliane modulari, Vol. 1, Edizioni Univ. "Docet", Roma, 1952; questi Rev. **14**, 859].

D. Gallarati (Genova).

Conforto, Fabio. Un'osservazione sopra le superficie abeliane impure di determinante primo. Boll. Un. Mat. Ital. (3) **8**, 2–6 (1953).

It is shown by means of a simple arithmetic argument that the existence of a singular relation for a Riemann matrix of genus 2 (belonging to a hyperelliptic surface of rank 1 carrying two elliptic pencils whose general elements intersect in an odd prime number of points) does not imply the existence of further singular relations with an arbitrarily given divisor.

O. F. G. Schilling (Chicago, Ill.).

Samuel, Pierre. Quelques tendances récentes de la géométrie algébrique. Enseignement Math. **39** (1942–1950), 180–191 (1953).

Segre, Beniamino. Sui corpi risolventi delle equazioni algebriche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 335–340 (1952).

Soit C une courbe de genre g définie sur un corps k de caractéristique 0. En utilisant les notions de diviseur de C rationnel sur k et de série linéaire rationnelle sur k, l'auteur démontre divers résultats arithmétiques relatifs aux sujets suivants: existence sur C de diviseurs positifs de degré donné (en particulier g) rationnels sur k; degrés des courbes planes birationnellement équivalentes à C sur k (avec, comme cas particulier, les résultats classiques de Noether-Hurwitz-Poincaré sur les courbes de genre 0); degrés sur k des corps où C possède des points rationnels. *P. Samuel.*

Northcott, D. G. On integrally closed geometric quotient rings and their extensions. Proc. London Math. Soc. (3) **2**, 385–405 (1952).

This paper is a continuation and amplification of àn earlier paper [Proc. Cambridge Philos. Soc. **47**, 662–667 (1951); these Rev. **13**, 489]. As an example of this general theory of extensions of geometric quotient rings and integrally closed quotient rings, Bézout's theorem is proved elegantly, making reference to a paper of A. Ostrowski [Abh. Math. Sem. Hamburg. Univ. **1**, 281–326 (1922)], which is used to set up the general counting of the total number of intersections. The author's arrangement of theorems brings out clearly analogies to similar results in the theory of discrete rank one valuations. For example, if Q is a regular geometric quotient ring (in the sense of Chevalley, Cohen, and Krull, or a quotient ring associated to a simple subvariety) with the quotient field F, and if F^* is a finite algebraic extension of F so that all regular quotient rings Q^* of F^* over Q are regular, then the integral closure of Q in F^* has a Q-module basis of $[F^*:F]$ elements. In the second part the author investigates to what extent one may relax the assumption of regularity and still develop a theory of unramified extension. An extension Q^*/Q is termed unramified if (i) $Q^*/\mathfrak{m}^*$ is separable over $Q/\mathfrak{m}$, and (ii) $[Q^*/\mathfrak{m}^*:Q/\mathfrak{m}]=[K^*:K]$, where $\mathfrak{m}^*$, $\mathfrak{m}$ are the maximal ideals of Q^*, Q, respectively, and K^*, K the quotient fields of the respective completions. For integrally closed rings Q passage to the completion is used to establish a necessary and sufficient condition for Q^*/Q to be unramified in terms of the Jacobian of a set of generating elements of K^*/K. Connections with the work of A. Weil, "Foundations of algebraic geometry" [Amer. Math. Soc. Colloq. Publ., v. 29, New York, 1946; these Rev. **9**, 303] on the counting of multiplicities are pointed out. Finally, by making reference to a result of Zariski (the completion of an integrally closed geometric quotient ring is integrally closed) it is shown that for integrally closed rings Q, Q^*/Q with separable $Q^*/\mathfrak{m}^*$ over $Q/\mathfrak{m}$, is unramified if and only if $Q^*\mathfrak{m}=\mathfrak{m}^*$, again a result which extends a well-known fact from the classical ramification theory. *O. F. G. Schilling.*

***Kähler, Erich. Sur la théorie des corps purement algébriques.** Deuxième Colloque de Géométrie Algébrique, Liège, 1952, pp. 69–82. Georges Thone, Liège; Masson & Cie, Paris, 1952. 375 Belgian francs; 2625 French francs.

This paper contains a series of definitions and statements, the latter without proofs, concerning the arithmetic of algebraic varieties which are finitely generated over a ring like the ring of integers. The terminology used by the author is rather unusual and tends to obscure the material content of his remarks. *O. F. G. Schilling.*

Differential Geometry

Bouligand, Georges. Sur les transformations de contact. C. R. Acad. Sci. Paris **236**, 1217–1219 (1953).

Les principaux résultats de cette note sont les suivants. (1) φ désignant une fonction de deux variables x et y, définie dans une région (ensemble ouvert connexe) τ du plan des (x, y), à dérivées premières continues, l'ensemble des plans tangents à la surface représentative $S: z=\varphi(x, y)$ dans l'espace euclidien E_3 n'admet pas de points intérieurs dans l'espace (tridimensionnel) des plans de E_3. (2) L'image Γ de S par la transformation $M=\tau(x, y, z, p, q)$ $(p=\partial z/\partial x, q=\partial z/\partial y)$ admettant des dérivées premières continues, est aplatie au sens suivant: l'image par τ d'une petite rondelle de S est incluse dans un cylindre droit C dont la hauteur est petite par rapport au rayon du cylindre droit maximum à base circulaire parallèle à C et inclu dans C. L'auteur donne de (1) une démonstration géométrique intuitive.

Chr. Pauc (Nantes).

Vidal Abascal, E. Concept of geometry and geometric space. Revision of the Erlangen program. Revista Mat. Hisp.-Amer. (4) **12**, 340–368 (1952). (Spanish)

This article is expository. It contains an account of the development of geometrical ideas from classical times up to the present, with special reference to the way in which Klein's Erlanger Programm relates to modern concepts in differential geometry. In Chapter I the author describes various notions of space, his discussion being based for the most part on the Veblen-Whitehead tract [The foundations of differential geometry, Cambridge, 1932]: a geometry is determined by a geometric object, and equivalence with respect to a pseudogroup is a natural generalization of Klein's group concept. In Chapter II he gives a brief account of Riemannian geometry (after the manner of Cartan), of the holonomic group, and of the affine and projective generalizations of Riemannian geometry. In Chapter III he discusses Lie groups, Klein spaces, and textile geometry. In the fourth and last chapter he deals with Klein and Cartan spaces from the point of view of the theory of fibre bundles, and a final note refers to various other types of space (Finsler, Kawaguchi, Hermitian, etc.).

H. S. Ruse (Leeds).

Ničе, Vilim. On Plücker's and some other conoids of 3rd and 4th order. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **276**, 117–124 (1949). (Serbo-Croatian)

Ničе, Vilim. Sur le conoïde de Plücker et certains conoïdes analogues du 3e et 4e ordre. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **5**, 39–40 (1952).

Summary of the paper listed above.

Ničе, Vilim. On the strophoidal and the space curve of 4th order on a sphere. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **276**, 109–116 (1949). (Serbo-Croatian)

Niče, Vilim. Sur la strophoïdale et la courbe gauche cyclique du 4e ordre. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **5**, 37–38 (1952).
Summary of the paper listed above.

Bereis, Rudolf. Über die Böschungslinien auf Drehquadriken. Monatsh. Math. **56**, 344–351 (1952).
A helix is a skew curve whose tangents form a constant angle with a straight line l. If it lies on a quadric of revolution about l, then its orthogonal projection parallel to l is a cycloidal curve Γ. The author presents a simple construction of the Γ's: Given a one-parametric family of conics C which is invariant under rotations about their common center 0, then the line-elements at the points of the C's parallel to their major (minor) axes are the tangents of a one-parametric family of Γ's which is likewise invariant under rotations about 0. The converse also holds true.
P. Scherk (Saskatoon, Sask.).

Wintner, Aurel. On the infinitesimal geometry of curves. Amer. J. Math. **75**, 241–259 (1953).
The object of this paper is to examine the theory of space curves with an eye to the differentiability conditions which are implied. Torsion is defined for curves of class C'' in a fashion believed new, but which occurs in the older literature [e.g., Duschek and Mayer, Lehrbuch der Differentialgeometrie, Bd. I, Teubner, Leipzig-Berlin, 1930, p. 43]. Using this definition various theorems are proved, such as: If a curve is of class C'', and is free of spherical points, the locus of the centers of its osculating spheres is a curve of class C'' possessing a non-vanishing continuous curvature and a non-vanishing continuous torsion. For a surface of class C^n ($n \geqq 3$) each evolute surface is locally of class C^{n-1}. For a curve of class C^n ($n \geqq 3$) where $\kappa > 0$, $\kappa' \neq 0$, the evolute is of class C^{n-1} locally. A curve of class C'' possessing continuous torsion is a curve on a unit sphere if and only if it is of class C''' and κ and τ satisfy: $\kappa \geqq 1$; $\pm\tau = \kappa'(\kappa^4 - \kappa^2)^{1/2}$ if $\kappa > 1$; $\pm\tau = \kappa''^{1/2}$ if $\kappa = 1$. Final results deal with the possibility of replacing the point coordinates of a plane curve by its line coordinates. *C. B. Allendoerfer* (Seattle, Wash.).

Lenz, Hanfried. Über kreistreue konforme Abbildungen zyklischer Flächen aufeinander und auf die Ebene. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1952**, 55–69 (1953).
The paper is concerned with "cylinder surfaces" which are generated by a one-parameter family of circles in Euclidean 3-space. The author examines the properties of those surfaces of this type which can be mapped conformally into the plane so that some of the following properties hold: (1) the generating circles go into circles; (2) the cross-ratio of the generating circles is equal to that of their images; (3) an arc of a generating circle goes into a circular arc. Similar questions are discussed concerning mappings of portions of these surfaces into a torus.
C. B. Allendoerfer (Seattle, Wash.).

Pan, T. K. A proof of a sufficient condition that two surfaces be applicable. Amer. Math. Monthly **60**, 318–319 (1953).

Ulčar, Jože. Über einen Satz aus der Theorie der affinen Einteilung der Flächen zweiter Ordnung. Bull. Soc. Math. Phys. Macédoine **3**, 9–13 (1952). (Macedonian. German summary)

Scharff, Heinrich. Zur Darstellung der Krümmungen einer Flächenkurve mit Pfaffschen Formen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1952**, 71–74 (1953).
Using the notation of E. Cartan the author obtains expressions for the geodesic torsion, normal curvature, and geodesic curvature of a curve on a surface.
C. B. Allendoerfer (Seattle, Wash.).

Kruppa, Erwin. Über die dualen Gegenstücke zum Meusnierschen und Eulerschen Satz der Flächentheorie. Rend. Circ. Mat. Palermo (2) **1**, 209–216 (1952).
Von Mannheim, Hostinsky, und Blaschke sind die klassischen Sätze von Meusnier und Euler dualisiert worden. So hat man z.B.: die einer nichtabwickelbaren Fläche ϕ umschriebenen Torsen, die eine Tangente t von ϕ als gemeinsame Erzeugende besitzen, haben längs t Krümmungskegel, deren Achsen sich in einem Punkt M der Flächennormalen des Berührungspunktes von t schneiden. Verf. gibt einen neuen Beweis der Blaschkeschen Formel, welche das duale Gegenstück der Eulerschen ist. Er zeigt weiter dass die bezüglichen Sätze auch in der Kurventheorie duale Gegenstücke haben. So hat man z.B.: alle Torsen Γ durch eine Raumkurve C, die eine Monosekante s von C als gemeinsame Erzeugende besitzen, haben längs s Krümmungskegel, die der Kugel umschrieben sind, die C und s in ihrem Schnittpunkt P berührt. Konstruktive Anwendungen.
O. Bottema (Delft).

Rembs, Eduard. Zur Verbiegung von Flächen im Grossen. Math. Z. **56**, 271–279 (1952).
Four distinct results concerning isometric deformations of surfaces are given. 1) A discussion of the isometric mapping of certain ellipsoids on one another is given on the basis of explicit formulas. 2) It is pointed out that the proof of Herglotz [Abh. Math. Sem. Hansischen Univ. **15**, 127–129 (1943); these Rev. **7**, 322] for the uniqueness of a closed convex surface with given line element can be extended readily to the case of bounded convex open surfaces provided that the spherical image of the surface is always the whole sphere except for the omission of a finite number of points. 3) Examples of analytic surfaces of revolution of genus zero are given which are not infinitesimally rigid. Such surfaces have, of course, zones of negative as well as positive Gauss curvature. 4) It is shown that a surface of positive curvature whose spherical image is a half-sphere is infinitesimally rigid provided that the deformed surface is one whose spherical image has the same property.
J. J. Stoker (New York, N. Y.).

Biran, Lutfi. Généralisation de deux formules de J. Liouville. Rev. Fac. Sci. Univ. Istanbul (A) **18**, 109–115 (1953). (Turkish summary)
Faisant usage des paramètres duaux pour l'étude des congruences de droites, et désignant par $(\bar{X})$ le vecteur dualistique d'un rayon quelconque, l'auteur étend à la théorie des congruences rectilignes certains résultats de la géométrie différentielle des surfaces, $d\bar{X}^2$ (carré de l'élément d'arc dualistique de la congruence) jouant le rôle de la première forme fondamentale de la surface, les lignes de courbure de cette dernière étant remplacées par les surfaces principales de la congruence. Après avoir retrouvé une formule bien connue d'Hamilton, il établit les formules donnant, au moyen des courbures dualistiques sphériques σ_1, σ_2 des deux surfaces principales issues d'un même rayon, les dérivées des vecteurs dualistiques unitaires qui inter-

viennent dans les calculs. Il applique ensuite ces formules à l'établissement de deux relations étendant à la théorie des congruences rectilignes les formules de Liouville donnant, au moyen des courbures géodesiques ρ_1, ρ_2 des lignes coordonnées supposées orthogonales, la courbure géodésique d'une courbe quelconque tracée sur la surface et la courbure totale de cette derniére. Dans les extensions obtenues les courbures dualistiques σ_1 et σ_2 jouent le rôle des courbures géodésiques ρ_1 et ρ_2. *P. Vincensini* (Marseille).

Berezina, L. Ya. Refraction of a congruence with real focal surfaces by means of an optical system. Uspehi Matem. Nauk (N.S.) **8**, no. 1(53), 111–113 (1953). (Russian)

The theorem of Malus is generalized in the following way. When a congruence of rays (r_0) in a homogeneous medium is refracted into a second congruence of rays (r_2) along an arbitrary boundary surface Σ, then

$$\frac{2\rho_0 \cot 2\varphi_0}{m_0^2-\rho_0^2}\cot i=\frac{2\rho_2 \cot 2\varphi_2}{m_2^2-\rho_2^2}\cot i'.$$

Here i is the angle of incidence, i' the angle of refraction, $2\rho_k$ the distance between the foci on ray r_k, $2\varphi_k$ the angle between the focal planes of r_k, m_k the distance from the center of r_k to the point of refraction. The expression $2\rho_k \cot 2\varphi_k$ is the so-called anormality of the congruence (Finikov). Simplifications occur when the plane of refraction is a principal plane of Σ, and when Σ is a plane. In this second case the anormalities of congruence (r_0) and (r_2) are to each other as $\tan^2 i$ to $\tan^2 i'$. *D. J. Struik.*

Čahtauri, A. I. On projective bending of a plane net. Soobščeniya Akad. Nauk Gruzin. SSR **11**, 531–532 (1950). (Russian)

If $x^\alpha=x^\alpha(u^1, u^2)$, $a_{ij}du^i du^j=0$ define a plane net, they determine an affine connection Γ^m_{ij}. It is shown that if there are two such nets, projectively deformable into one another, it is necessary and sufficient that

$$^1\Gamma^m_{ij}=\Gamma^m_{ij}+p_i\delta^m_j+p_j\delta^m_i+\mu^m a_{ij}.$$

The author proves that if the two nets form a configuration of Laplace, $\mu^m=-\bar{a}^{ms}p_s$, where p_s is a gradient.

M. S. Knebelman (Pullman, Wash.).

Ströher, Wolfgang. Zur projektiven Differentialgeometrie ebener Kurven. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **160**, 199–207 (1951).

Let T_2 be a neighboring point of a point T_1 on an arc k without singularities in a projective plane, t_1, t_2 and c_1, c_2 the tangents and osculating conics of k at T_1 and T_2, and P_1 and P_2 the poles of the line T_1T_2 with respect to c_1 and c_2, respectively. Then, as T_2 approaches T_1, the limiting position of the line P_1P_2 is the projective normal of k at T_1. The projective arc element and curvature of k at T_1 are the limiting values, as T_2 approaches T_1, of two expressions in terms of the three invariants of Inzinger [Monatsh. Math. **56**, 38–48 (1952); these Rev. **14**, 85] for the two curvilinear elements of the third order consisting of c_1, t_1 and c_2, t_2 respectively. *C. C. Hsiung* (Bethlehem, Pa.).

Ancochea, G. Géométrie différentielle des singularités des courbes de l'espace projectif. Rend. Circ. Mat. Palermo (2) **1**, 217–239 (1952).

Il n'y a pas longtemps que l'auteur a donné [Abh. Math. Sem. Univ. Hamburg **18**, 1–13 (1952); ces Rev. **14**, 687] la théorie complète des singularités pour les courbes planes; il y ajoute maintenant celle pour les courbes du S_n, suivant une méthode analogue. Il commence à considérer les branches singulières de l'espace affine à trois dimensions. Les coordonnées d'une branche d'ordre m, de rang $n-m$, et de classe $p-n$ seront

$$x=\alpha_m t^m+\alpha_{m+1}t^{m+1}+\cdots,\quad y=\beta_n t^n+\beta_{n+1}t^{n+1}+\cdots,$$
$$z=\gamma_p t^p+\gamma_{p+1}t^{p+1}+\cdots$$

où $z=0$ est le plan osculateur, $y=z=0$ la tangente. En définissant la binormale affine et la normale principale affine, après un changement de paramètre et après une transformation de coordonnées on arrive à

$$x=t^m+\mu_{m+1}t^{m+1}+\cdots,\quad y=t^n+\nu_{n+1}t^{n+1}\cdots,\quad z=t^p.$$

Si s est le plus petit entier pour lequel $\nu_{n+s}\neq 0$, il est possible d'obtenir $\nu_{n+s}=1$. On a ainsi fixé le point unité du répère intrinsèque affine. Les autres valeurs μ, ν sont des invariants absolus et constituent un système complet.

En montant à la géométrie projective l'auteur réussit à fixer un plan covariant, ne pas passant par O, qui peut être choisi comme le plan à l'infini, et de ce façon il réduit l'étude des invariants projectifs au cas précédant. Il considère en détail le cas special $p=n+m=3m$, qui comprend celui des points ordinaires. Pour les cas $m=2$, $n=3$, $p=4$ (point de rebroussement), $m=1$, $n=3$, $p=4$ (point d'inflexion), $m=1$, $n=2$, $p=4$ (point ordinaire à plan osculateur stationnaire) il donne une interprétation géométrique des éléments du repère intrinsèque.

Il finit par considérer l'étude des singularités dans l'espace projectif S_n. *O. Bottema* (Delft).

Tsuboko, Matsuji. On a method of plunging of R_2 with symmetric projective connection into a four-dimensional projective space S_4. Tensor (N.S.) **2**, 162–168 (1952).

Let R_2 be a two-dimensional space with a symmetric projective connection, and V_2 a surface in a four-dimensional projective space S_4 defined by, with respect to a frame $[A_0A_1\cdots A_4]$ associated with a moving point $A_0(x^1, x^2)$ of V_2,

$$dA_\sigma=\Gamma^\tau_{\sigma i}dx^i A_\tau \quad (i=1, 2;\ \sigma, \tau=0, 1, \cdots, 4),$$

with the conditions $\Gamma^3_{11}=\Gamma^3_{22}=\Gamma^4_{12}=1$, $\Gamma^3_{12}=\Gamma^4_{11}=\Gamma^4_{22}=0$. It is shown that R_2 can be plunged into S_4 in such a way that R_2 becomes V_2 and a net of curves arbitrarily given on R_2 becomes a net of curves on V_2, whose osculating planes at A_0 intersect in a line through A_0. Moreover, there exist at least a two-dimensional projective space and a normal space among the two-dimensional spaces with a symmetric projective connection which can be plunged into S_4 as surfaces projectively coinciding with V_2. *C. C. Hsiung.*

Suguri, Tsuneo. Theory of invariants in the geometry of paths. J. Math. Soc. Japan **4**, 231–268 (1952).

This paper is a more elaborate discussion of a subject dealt with previously by the same author [Proc. Japan Acad. **26**, nos. 2–5, 97–103, 104–106 (1950); these Rev. **14**, 409]. See Rev. 15 233 *A. Nijenhuis* (Princeton, N. J.).

Maruyama, Takaharu. Some properties on geometry in complex space. II. J. Gakugei Coll. Tokushima Univ. **2**, i–v (1952).

[For part I see J. Sci. Gakugei Fac. Tokushima Univ. **1**, 31–35 (1950); these Rev. **13**, 487.] It is said that the equations $A_i x^i=0$, $B_i x^i=0$ of a plane I, of the four-dimensional euclidean space, are in normal form when each equation is in normal form and its corresponding hyperplanes are orthogonal. The plane I is said to be isoclinic to the plane

$A_i'x^i=0$, $B_i'x^i=0$, when, if the equations of the two planes are in normal form and if the intersection point of the two is taken as the origin of coordinates, it is true, using the Gauss notation, that $[AA']=[BB']$, $[AB']=-[A'B]$ or $[AA']=-[BB']$, $[AB']=[A'B]$. The transitivity of the isoclinic relation is proved and other elementary metrical properties are studied. *P. Abellanas* (Madrid).

Fernández, Germán. A theorem on curves on a hypersurface in a Riemann space. Math. Notae **12–13**, 38–47 (1952). (Spanish)

The author proves the following theorem: Let V_{n+1} be a Riemannian space of $n+1$ dimensions, and P a point of a hypersurface V_n of V_{n+1}; then, all the curves of V_n going through P and having at P the same Frenet-Serret n-hedron, have also in common at P the normal curvature, the $n-1$ geodesic torsions, and $2n$ quantities, involving the normal curvatures of V_n and the curvature tensor of V_{n+1}. The author's method is purely analytic and follows very closely the method of E. Cartan [Leçons sur la géométrie des espaces de Riemann, 2d ed., Gauthier-Villars, Paris, 1946; these Rev. **8**, 602], to whom the theorem was due for $n=2$. *V. Dalla Volta* (Rome).

Matsumoto, Makoto. Riemann spaces of recurrent and separated curvature and their imbedding. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 175–188 (1952).

A Riemann space V_n is said to be of type τ (>1) if the matrix $\|R_{(abc)(d)}\|$ is of rank τ. Here the set of indices (abc) indicates the rows, and (d) indicates the columns. The author shows that if V_n is of type 2, then there exist H_{ij} satisfying $R_{ijkl}=H_{ik}H_{jl}-H_{il}H_{jk}$ if and only if the rank of $\|R_{(ab)(cd)}\|$ is one. The conditions that H_{ik} satisfy the Codazzi equations are only partially obtained. The proof incidentally shows that $R_{ijkl}=eS_{ij}S_{kl}$, where $\|S_{ij}\|$ is of rank two. A space with this property is called of "separated curvature". It is proved that such a space is of recurrent curvature [H. S. Ruse, J. London Math. Soc. **21**, 243–247 (1947); these Rev. **9**, 102] if and only if there exists a σ such that $T_{ij}=\sigma S_{ij}$ satisfies $T_{ij,k}=0$. A tensor X_{ij} is called "semi-Codazzi" if there exists a σ such that $Y_{ij}=\sigma X_{ij}$ satisfies $Y_{ij,k}=Y_{ik,j}$. Necessary and sufficient conditions that X_{ij} be semi-Codazzi are obtained. *C. B. Allendoerfer* (Seattle, Wash.).

Tomonaga, Yasurô. Betti numbers and exact differential forms. J. Math. Soc. Japan **4**, 269–278 (1952).

The theory of harmonic integrals on a compact Riemannian manifold is used to obtain inequalities for the Betti numbers when the curvature tensor has certain properties, or the existence of certain types of tensors on the manifold is known. The conditions (some of which could not easily be verified in a given case) are too complicated to be quoted here. *W. V. D. Hodge* (Cambridge, England).

Blanchard, André. Variétés kählériennes et espaces fibrés. C. R. Acad. Sci. Paris **234**, 284–286 (1952).

Let V denote a complex analytic variety, E a complex analytic principal fibre space with base V and fibre a complex abelian Lie group $G\approx C^n/\Pi$ where Π is a discrete subgroup of the complex n-space C^n. In terms of a locally finite open covering $\{U_j\}$ of V, E is determined by a system $\{f_{jk}\}$ of functions f_{jk}: $U_j\cap U_k\to G$ which satisfy $f_{jk}+f_{kl}+f_{lj}=0$ in $U_j\cap U_k\cap U_l$. The fibre space E is said to admit an invariant analytic connection if there exists a system $\{h_k\}$ of holomorphic 1-forms h_j with coefficients in C^n defined respectively in U_j such that $h_j-h_k=df_{jk}$ in $U_j\cap U_k$. It is always possible to define a differentiable connection by means of forms h_j of type $(1,0)$ and class C^∞, and we let σ be the 2-form with coefficients in C^∞ which is equal to dh_j in U_j. If V is compact and Kählerian, it is possible to choose the h_j such that they are harmonic and, in this case, the author proves that an invariant analytic connection exists if and only if σ is of type $(2,0)$. If there is a system $\{h_j\}$ of closed forms h_j of type $(1,0)$ satisfying $h_j-h_k=df_{jk}$ in $U_j\cap U_k$, E is said to possess an integrable connection. If G is a complex torus, the author shows that E is Kählerian if and only if V is Kählerian and E has an integrable connection. *D. C. Spencer* (Princeton, N. J.).

Duff, G. F. D. Harmonic p-tensors on normal hyperbolic Riemannian spaces. Canadian J. Math. **5**, 57–80 (1953).

Soit M un espace de Riemann orientable, de dimension m, muni d'une métrique hyperbolique normale, i.e., après transformation en une somme de carrés, le ds^2 contient un terme positif et $m-1$ termes négatifs. On appelle déplacements dans le temps (resp. l'espace) les déplacements pour lesquels $ds^2>0$ (resp. $ds^2<0$). De même, on distingue la coordonnée temps t. On se propose de résoudre le problème de Cauchy pour l'équation de Laplace-Beltrami relative aux tenseurs covariants antisymétriques de rang p (formes différentielles ϕ de degré $p\leqq m$). La surface initiale S est supposée spatiale (définie par $t=0$), compacte et suffisamment différentiable. Soit R_S la région formée des points P en lesquels $t\geqq 0$ et tels que les cônes caractéristiques rétrogrades de sommet P ainsi que la partie de S qu'ils interceptent bordent une région simplement connexe de M. Soit $t\alpha$ (resp. $n\alpha$) la composante tangentielle (resp. normale) de la forme différentielle α en un point de S. La donnée initiale se compose d'une p-forme ξ sur S et de $nd\xi$ et $t\delta\xi$. Alors, la solution du problème de Cauchy pour l'équation sans second membre est unique dans R_S. Pour la construction, sur R_S, de la solution d'une équation avec second membre, on utilise la méthode donnée par M. Riesz dans le cas scalaire [Acta Math. **81**, 1–223 (1949); ces Rev. **10**, 713]. Il en résulte, sur R_S, la solution des équations $d\phi=\rho$, $\delta\phi=\sigma$, la donnée initiale étant une forme ξ sur S qui satisfait à $td\xi=t\rho$, $n\delta\xi=n\sigma$ sur S. Les conditions de différentiabilité des données sont précisées. Les démonstrations supposent la métrique analytique mais cette restriction peut être levée. Parmi des cas particuliers, on examine les équations du champ électromagnétique. Dans le cas hyperbolique, on remarque que les conditions aux limites assurant l'unicité de la solution sont plus nombreuses que dans le cas elliptique [G. F. D. Duff et D. C. Spencer, Ann. of Math. (2) **56**, 128–156 (1952); ces Rev. **13**, 987], par contre aucune condition sur les périodes n'est nécessaire, la région R_S n'ayant pas de cycles relatifs (mod S) non homologues à zéro et les cycles absolus de R_S étant homologues à des cycles de S. *P. Dolbeault.*

Duff, G. F. D. Boundary value problems associated with the tensor Laplace equation. Canadian J. Math. **5**, 196–210 (1953).

This paper contains proofs of the existence of a form ϕ which is harmonic in an orientable Riemannian manifold (C^∞) M of dimension n with a regular submanifold B of dimension $n-1$ as boundary, (a) when the tangential and normal components of ϕ are given on B, (b) when the tangential compact of $\delta\phi$ and the normal component of $d\phi$ are given on B, and (c) under certain mixed boundary conditions. (b) requires that the condition $f_B(\tau_n{}^*d\phi-\delta\phi_n{}^*\tau)=0$ be satisfied for every harmonic field τ in M. The theorems are established by solving certain integral equations on B,

and the proofs given of (a) and (b) do not require the assumption of the uniqueness theorem, as in a previous proof [cf. Duff and Spencer, Proc. Nat. Acad. Sci. U. S. A. **37**, 614–619 (1951); these Rev. **13**, 385]. The paper concludes with a brief examination of an associated eigenvalue problem. *W. V. D. Hodge* (Cambridge, England).

Jongmans, F. Relations entre les périodes des formes harmoniques attachées à une variété kählérienne. Bull. Soc. Roy. Sci. Liège **21**, 18–23 (1952).

Let V be a compact Kählerian variety, and let ω be the fundamental harmonic form of type (1, 1) associated with the metric. By a well-known theorem of Hodge, any harmonic differential form of degree p and of type $(p-h, h)$ has the decomposition $\phi=\phi_0+\omega\wedge\phi_1+\cdots+\omega^q\wedge\phi_q$ where $q=[p/2]$ and ϕ_j is an effective harmonic form of type $(p-h-j, h-j)$. We say that a form ϕ is of class k if $\phi=\omega^k\wedge\psi$ where ψ is effective, and we denote by $H^p{}_{h,k}$ the complex vector space of harmonic forms of degree p, type $(p-h, h)$, and class k. Let $b^p{}_{h,k}$ denote the dimension of $H^p{}_{h,k}$, $b^p=\sum_{h,k}b^p{}_{h,k}$ the pth Betti number of V, and let $\{C_\mu{}^p\}$ be a rational homology base of V of dimension p. If $\phi\in H^p{}_{h,k}$, $p\neq 2h$, and if P_μ $(\mu=1, 2, \cdots, b^p)$ are the periods of ϕ with respect to the $C_\mu{}^p$, the author shows that there are $b^p-2b^p{}_{h,k}$ linearly independent relations $\sum_\mu B_\mu{}^i P_\mu=0$ where the $B_\mu{}^i$ $(\mu=1, 2, \cdots, b^p;\ j=1, 2, \cdots, b^p-2b^p{}_{h,k})$ are real coefficients independent of the choice of ϕ, and that there is one and only one form $\phi\in H^p{}_{h,k}$ with $\operatorname{Re} P_\mu=\theta_\mu$ where $\sum_\mu B_\mu{}^i\theta_\mu=0$. *D. C. Spencer* (Princeton, N. J.).

Jongmans, F. Les variétés kählériennes. Bull. Soc. Roy. Sci. Liège **21**, 345–363 (1952).

Exposé de synthèse de la théorie des formes différentielles régulières sur les variétés kählériennes. On rappelle: 1) les résultats d'Eckmann et Guggenheimer [Eckmann et Guggenheimer, C. R. Acad. Sci. Paris **229**, 464–466, 489–491, 503–505 (1949); ces Rev. **11**, 212; Eckmann, ibid. **229**, 577–579 (1949); ces Rev. **11**, 212] en distinguant les théorèmes de décomposition valables sur toute variété hermitienne des théorèmes de décomposition des formes harmoniques valables seulement sur les variétés kählériennes; 2) les résultats de l'auteur sur les diviseurs de zéro de l'anneau de cohomologie [ibid. **233**, 1254–1256 (1951); ces Rev. **13**, 584] et sur les périodes des formes pures de classe déterminée des variétés kählériennes compactes [voir l'analyse ci-dessus]. Le formalisme adopté est celui introduit par A. Weil [Comment. Math. Helv. **20**, 110–116 (1947); ces Rev. **9**, 65]. *P. Dolbeault* (Paris).

Bochner, S., and Yano, K. Tensor-fields in non-symmetric connections. Ann. of Math. (2) **56**, 504–519 (1952).

Les auteurs poursuivent, dans le cas des connexions euclidiennes nonsymétriques, les études entreprises dans le cas riemannien par Bochner [Ann. of Math. (2) **49**, 379–390 (1948); **50**, 77–93 (1949); ces Rev. **9**, 618; **10**, 571], Lichernowicz [C. R. Acad. Sci. Paris **226**, 1678–1680 (1948); Proc. Internat. Congress Math. Cambridge, Mass., 1950, v. 2, Amer. Math. Soc., Providence, R. I., 1952, pp. 216–223; ces Rev. **9**, 618; **13**, 492], et Yano [Ann. of Math. (2) **53**, 38–45, 328–347 (1952); ces Rev. **13**, 689; **14**, 88]. La méthode générale reste la même. Dans une première partie, les formules fondamentals relatives à une connexion euclidienne $E^i{}_{jk}$, à son tenseur de torsion $S^i{}_{jk}$, à son tenseur de courbure $E^i{}_{j,kl}$, sont rappelées et deux cas particuliers, intéressants pour la suite, mis en évidence: celui où le tenseur covariant de torsion est complètement antisymétrique et celui où le vecteur $S^i{}_j$, est nul (qui englobe le précédent). Un exemple typique de la première circonstance est pris, celui de l'espace d'un groupe compact semi-simple, pour lequel les résultats classique de Cartan-Schouten sont rappelés. Dans une second partie, les auteurs, supposant désormais compacte la variété V_n à connexion euclidienne, étudient les vecteurs pseudo-harmoniques et les pseudo-vecteurs de Killing, qui sont définis en remplaçant dans les formules riemanniennes habituelles la connexion symétrique classique par la connexion $E^i{}_{jk}$. Des résultats variés sont obtenus, dont les suivants sont typiques: si V_n admet un tenseur de torsion antisymétrique, les pseudo-vecteurs de Killing et les vecteurs de Killing de la métrique coïncident; si de plus le tenseur de courbure contracté E_{ij} (pour la connexion E) est aussi antisymétrique, tout vecteur pseudo-harmonique admet une dérivée covariante nulle dans la connexion E et est vecteur de Killing de la métrique. Les résultats classiques relatifs aux variétés de groupe sont retrouvés par cette voie. Enfin, dans une troisième partie, les auteurs généralisent certains résultats aux p-formes qui sont soit pseudo-harmoniques, soit pseudo-formes de Killing. Un énoncé dû en fait à Lichnerowicz [voir travaux cités] relatifs aux nombres de Betti d'une variété riemannienne est généralisé sous forme d'un énoncé relatif aux p-formes pseudo-harmoniques [le corollaire 2 du théorème 9 semble aussi étroitement lié géométriquement au dernier résultat de la note citée du rapporteur]. Un essai d'extension un peu formelle termine ce travail. *A. Lichnerowicz* (Paris).

Moór, A. Finslersche Räume mit algebraischen Grundfunktionen. Publ. Math. Debrecen **2**, 178–190 (1952).

On making use of the algebraic invariants and covariants of a quartic form: $f=a_0\dot{x}^4+4a_1\dot{x}^3\dot{y}+6a_2\dot{x}^2\dot{y}^2+4a_3\dot{x}\dot{y}^3+a_4\dot{y}^4$, the author takes two-dimensional Finsler spaces with the fundamental function: $L=f^{1/4}$ into consideration. After the explanation of the invariants

$$i=a_0a_4-4a_1a_3+3a_2^2,\quad j=\begin{vmatrix}a_0 & a_1 & a_2\\ a_1 & a_2 & a_3\\ a_2 & a_3 & a_4\end{vmatrix},$$

discriminant $D=i^3-27j^2$ and covariants

$$H_4\text{ (Hessian form)},\quad T_4=\frac{1}{8}\left(\frac{\partial f}{\partial\dot{x}}\frac{\partial H_4}{\partial\dot{y}}-\frac{\partial f}{\partial\dot{y}}\frac{\partial H_4}{\partial\dot{x}}\right)$$

of the form f, the fundamental function L is transformed into certain simple types in every one of the cases: (1) $D=0$, $i=j=0$; (2) $D=0$, $i\neq 0$, $j\neq 0$; (3) $D\neq 0$. The Berwald principal scalar [J. Reine Angew. Math. **156**, 191–210, 211–222 (1927)] and the curvature scalar are calculated in the case (1). It is also shown that some special types of the spaces can be characterized by use of these invariants. In the last section the equivalent problem of Finsler spaces with the fundamental function $F=f^{1/4}$ is solved by means of the invariants i, j and of the principal curvature tensor and the torsion tensor. The use of these invariants makes the conditions of equivalence essentially simpler than those of the affinely connected line-element manifolds of Varga [Publ. Math. Debrecen **1**, 7–17 (1949); these Rev. **11**, 134]. Reviewer's note: This theory can be simplified if one uses the theory of Liber [Trudy Sem. Vektor. Tenzor. Analizu **9**, 319–350 (1952); these Rev. **14**, 688]. *A. Kawaguchi.*

Pastori, Maria. Applicazioni di calcolo tensoriale. Matematiche, Catania **7**, 21–54 (1952).

Three expository lectures given in May, 1952.

NUMERICAL AND GRAPHICAL METHODS

***Table of Arctan x.** National Bureau of Standards Applied Mathematics Series, No. 26, U. S. Government Printing Office, Washington, D. C., 1953. xiii+170 pp. $1.75.

Reprinted from the first edition [1942; these Rev. 4, 89]; known misprints have been corrected.

***Tables of Bessel-Clifford functions of orders zero and one.** National Bureau of Standards Applied Mathematics Series, no. 28. U. S. Government Printing Office, Washington, D. C. x+72 pp. $0.45.

This useful booklet gives tables of Bessel functions of argument $2\sqrt{x}$, commonly known as Bessel-Clifford functions. The functions $J_0(2\sqrt{x})$ and $Y_0(2\sqrt{x})$ are given to 8 decimals, whilst $J_1(2\sqrt{x})/\sqrt{x}$ and $Y_1(2\sqrt{x})/\sqrt{x}$ are given to 8 decimals for $x<25$, then to 9 decimals, in each case for $x=0(.02)1.5(.05)3(.1)13(.2)45(.5)115(1)410$. The modified functions $I_0(2\sqrt{x})$, $K_0(2\sqrt{x})$, $I_1(2\sqrt{x})/x$, and $K_1(2\sqrt{x})/x$ are tabulated for $x=0(.02)1.5(.05)6.2$, and the auxiliary functions $e^{-2\sqrt{x}}I_0(2\sqrt{x})$, $e^{-2\sqrt{x}}I_1(2\sqrt{x})/\sqrt{x}$, $e^{2\sqrt{x}}K_0(2\sqrt{x})$, and $e^{2\sqrt{x}}K_1(2\sqrt{x})/\sqrt{x}$ for

$$x=6.2(.1)13(.2)36(.5)115(1)160(5)410.$$

The number of decimals varies from 6 to 9, but 7 significant figures are always available, sometimes 8 or 9.

The introduction gives some useful formulae and indicates applications. Second differences, modified to give maximum accuracy, are given nearly everywhere as an aid to interpolation, although it seems probable that the main usefulness of the tables will be in cases where the need may be confined to tabular values, since tables of the ordinary Bessel functions meet most needs requiring interpolation. However, the region with $x>100$ covers new ground and considerably enhances the importance of the tables.

J. C. P. Miller (Cambridge, England).

Struve function of order three-halves. J. Research Nat. Bur. Standards **50**, 21–29 (1953).

This paper gives values of the function

$$h_{3/2}(x)=(2\pi/x)^{1/2}H_{3/2}(x)=\left(1+\frac{3}{x^2}\right)-\frac{2}{3}\left(\sin x+\frac{\cos x}{x}\right)$$

to 10 decimals for $x=0(.02)15$. Modified second differences are given as an aid to interpolation.

The tables are slightly marred by the unnecessary and unusual convention adopted of putting the sign for the difference after the numerical value and by the letter C, used to indicate modification; it would have been better to modify all differences and say that this had been done. The suggestion that mechanical tabulation depended on these modifications is unacceptable. *J. C. P. Miller.*

Cherry, T. M. Tables and approximate formulae for hypergeometric functions, of high order, occurring in gas-flow theory. Proc. Roy. Soc. London. Ser. A. **217**, 222–234 (1953).

In this paper the author discusses those hypergeometric functions which occur in the hodograph method for calculating trans-sonic flow. The adiabatic exponent γ is taken as 1.4, a_ν and b_ν are determined by

$$a_\nu+b_\nu=\nu-\frac{1}{\gamma-1},\quad a_\nu b_\nu=-\frac{\nu(\nu+1)}{2(\gamma-1)},\quad a_\nu>b_\nu,$$

and the functions in question are

$$\chi_\nu(\tau)=\tau^{b_\nu}F(\nu-a_\nu,\nu-b_\nu;\nu+1;\tau)$$
$$\psi_\nu(\tau)=\tau^{b_\nu}F(a_\nu,b_\nu;\nu+1;\tau).$$

The point $\tau_s=(\gamma-1)/(\gamma+1)=1/6$ is a transition point of the differential equation satisfied by χ_ν and ψ_ν, and no asymptotic representation in terms of elementary functions is available in an interval including this point. Asymptotic formulas in terms of Airy functions were given by the author [same Proc. **202**, 507–522 (1950); these Rev. **12**, 257].

The functions $\chi_{-\nu}$, $\psi_{-\nu}$ are singular when ν is a positive integer, and in this case modified functions $\chi y_\nu(\tau)$, $\psi y_\nu(\tau)$ are introduced (which correspond to Bessel functions of the second kind). In the present paper the author tabulates certain slowly varying auxiliary functions which permit computation of $\chi_\nu(\tau)$, $\psi_\nu(\tau)$, $\chi y_\nu(\tau)$, $\psi y_\nu(\tau)$, $\chi_\nu'(\tau)$, $\chi y_\nu'(\tau)$ for $\tau=.08(.02).30$, $\nu=10.5(1)30.5$. The entries are given to 6S for $\nu\leqq 20.5$, to 4S for $\nu>20.5$. The last figure may occasionally be two units in error but the author states that a complete check by means of certain identities suggests that there are very few 2-figure errors.

For $\nu>30.5$, the author gives certain approximations derived by "fitting" an elementary approximation to the Airy function for both large and small variables. The approximations are rational functions of ν, and the numerical coefficients in these are tabulated for $\tau=.08(.02).3$.

A. Erdélyi (Pasadena, Calif.).

Keller, Geoffrey, and Fenwick, Mary. Tabulation of the incomplete Fermi-Dirac functions. Astrophys. J. **117**, 437–446 (1953).

The incomplete Fermi-Dirac function is "needed to determine the density of free electrons within the bound orbits of atoms at high temperatures and pressures". The function is defined as

$$F(\eta,u)=\int_0^u(e^{x-\eta}+1)^{-1}x^{1/2}dx.$$

The author gives an approximation in terms of the probability integral for $\eta<-2$, and approximations in terms of the functions

$$G_0(v)=\int_v^\infty(e^y+1)^{-1}dy,\quad G_n(v)=c_n\int_0^v y^n(e^y+1)^{-1}dy,$$

$$c_1=\tfrac{1}{2},\quad c_2=\tfrac{1}{8},\quad c_3=\tfrac{1}{16},\ \cdots$$

for $\eta>10$. They tabulate $F(\eta,u)$ to 3, 4, or 5S for $\eta=-2(.5)10$, $u=0(.2)1(.1)19.5$, and $G_n(v)$ to 3 or 4D for $n=0(1)3$, $v=0(.1)10$. *A. Erdélyi* (Pasadena, Calif.).

Tables of complete downwash due to a rectangular vortex. Computed by the staff of the Mathematics Division, N.P.L. Preface by V. M. Falkner. Ministry of Supply (London), Aeronaut. Res. Council, Rep. and Memoranda no. **2461**, 18 pp. (1953).

Comprehensive table of the quantity

$$F=-\frac{1}{x}\left[\frac{y+1}{(x^2+(y+1)^2)^{1/2}}-\frac{y-1}{(x^2+(y-1)^2)^{1/2}}\right]$$
$$-\frac{1}{y-1}\left[1-\frac{x}{(x^2+(y-1)^2)^{1/2}}\right]+\frac{1}{y+1}\left[1-\frac{x}{(x^2+(y+1)^2)^{1/2}}\right]$$

for various values of x and y. *E. Reissner.*

Mayot, Marcel. Tables de fonctions intervenant dans le calcul des corrections de diffusion dans la photomètrie de la lumière du ciel nocturne. Ann. Astrophysique **15**, 374–382 (1952).

By a change of the variable of integration the author transforms $\Phi_i(\tau, \alpha)$, $i=0, 1, 2$, to the form

$$\Phi_i(\tau, \alpha)=\frac{\beta^i}{\alpha}F_{i+1}(\lambda, \beta), \quad \beta^2=\alpha^{-2}-1, \quad \lambda=\tau/\beta,$$

where $F_k(\lambda, \beta)=\int_\beta^\infty u^{-k}(1+u^2)^{-1/2}e^{-\lambda u}du$. He tabulates $\Phi_i(\tau, \alpha)$ to 4D for $i=0, 1, 2$, $\tau=0(.05).7$, $\alpha=R/(R+H)$, $R=6370$, $H=0, 50, 100, 200, 400, 800, \infty$. Some series expansions are also given and the method of computation is described.

A. Erdélyi (Pasadena, Calif.).

***Sadosky, Manuel. Cálculo numérico y gráfico. [Numerical and graphical calculation.]** Ediciones Librería del Colegio, Buenos Aires, 1952. 347 pp. 54 Argentine pesos.

An elementary text with chapter headings as follows: Numerical approximations, Scales, Logarithmic graphs, Slide rules, Nomography, Linear systems, Numerical solution of equations, Interpolation, Numerical differentiation and integration, Graphical and mechanical integration, Approximate integration of differential equations. Finally there is a twelve-page appendix entitled Evolution of mechanical and automatic calculation. In illustration of the level, Chapter VII gives the methods of regula falsi, Newton (with a section on iterative processes in general), Horner, and Graeffe, with discussions of synthetic division, Descartes's rule of signs, and Sturm's theorem. There are numerous examples, many worked out in detail. *A. S. Householder.*

***Yakovlev, K. P. Matematičeskaya obrabotka rezul'tatov izmerenii. [Mathematical treatment of the results of measurements.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 388 pp. 10.80 rubles.

This book presents in direct and elementary manner the theory of errors, adjustment of observations, least squares and allied topics. The explanations are detailed and are well illustrated with examples and drawings. The scope of the text may be seen from the following summary of topics. 1. Approximate quantities and their errors; round-off; rules for round-off; significant figures; classification of approximate quantities. 2. Basic operations with approximate numbers; orders of small quantities; formulas for approximate calculation; the six operations of arithmetic. 3. Errors of functions and general theory of errors; errors of functions due to errors of the independent variables; errors of the variables due to errors in the functions; optimum conditions for measurement. 4. Distribution of random errors; random phenomena and their classification; probability; theorems of addition and multiplication of probabilities; the probability integral for random errors. 5. Indices of accuracy of measurement; arithmetic mean; quadratic mean; probable mean; error of the arithmetic mean; weighted measurements; adjustment of observations. 6. Graphical representation of results of measurement; graphical differentiation and integration; functional scales (logarithmic, power, projective, etc.); coordinate nets (log-log paper, log paper, etc.). 7. Nomography; network nomograms; nomograms using aligned points; Z-nomograms. 8. Interpolation; linear and graphical interpolation; equally spaced points; Newton's formula; unequally spaced points; accuracy of interpolation. 9. Harmonic analysis; periodic processes; set-up for 12 ordinates; technical procedures. 10. Empirical formulas; graphical methods; interpolation formulas; method of means; method of least squares. Appendix: Formulas for approximate calculation; absolute and relative errors of important functions; table of the probability integral to five places at intervals of 0.01. *W. E. Milne.*

Guest, P. G. Note on the fitting of polynomials to equally-spaced observations. J. Math. Physics **32**, 68–71 (1953).

Given the polynomial

$$y=a_0+a_1\epsilon+a_2\epsilon^2+\cdots+a_r\epsilon^r,$$

where $\epsilon=(x-\bar{x})$, a simple method of determining the a_i is given, using power moments $S(y\epsilon^i)$. With the aid of a few formulas and a model computing form, the author computes the usual orthogonal coefficients and then the a_i in terms of these orthogonal coefficients, for $r\leq 4$. This method has the advantages of the orthogonal polynomial method without requiring its extensive tables. Some comparisons of computing times are needed. *R. L. Anderson.*

Vernotte, Pierre. La dérivation successive des courbes expérimentales, et son lissage. C. R. Acad. Sci. Paris **236**, 1737–1739 (1953).

Davidenko, D. F. On a new method of numerical solution of systems of nonlinear equations. Doklady Akad. Nauk SSSR (N.S.) **88**, 601–602 (1953). (Russian)

Given a system $f_k(x_1, \cdots, x_n, \lambda)=0$ of n equations whose solutions is required for $\lambda=\lambda_1$ and known for $\lambda=\lambda_0$, while in a suitable region the Jacobian exists and is non-null, the author proposes to write the system

$$\sum(\partial f_k/\partial x_\tau)dx_\tau/d\lambda=-\partial f_k/\partial\lambda,$$

solve for $dx_\tau/d\lambda$, and integrate numerically from the known solution at λ_0. *A. S. Householder* (Oak Ridge, Tenn.).

de la Garza, A. Error bounds on approximate solutions to systems of linear algebraic equations. Math. Tables and Other Aids to Computation **7**, 81–84 (1953).

Define $\alpha(A)$ to be the matrix of absolute values $|a_{ij}|$; $\alpha(A)\leqq\alpha(B)$ to mean $|a_{ij}|\leqq|b_{ij}|$ for all i and j; $D=I-GA$; and suppose $\lim_{p=\infty}D^p=0$. Let $A\xi=y$, $x=Gy$, $\epsilon=\xi-x$. If γ is a vector of positive elements, there exists a scalar k such that $\alpha(D)\gamma\leqq k\gamma$. Assume $k<1$. If e_j is the jth column of I, there is a scalar f_j depending on γ such that $\alpha(D)e_j\leqq f_j\gamma$. Then

$$\alpha\left(\sum_1^\infty D^\lambda e_j\right)\leqq f_j\gamma/(1-k), \quad \text{and} \quad \alpha(\epsilon)\leqq[I+\alpha(\textstyle\sum D^\lambda)]\alpha(Dx).$$

Specific bounds are obtained by making specific choices of f_j and k. *A. S. Householder* (Oak Ridge, Tenn.).

Bejarano, Gabriel G., and Rosenblatt, Bruce R. A solution of simultaneous linear equations and matrix inversion with high speed computing devices. Math. Tables and Other Aids to Computation **7**, 77–81 (1953).

Jeffreys, Harold. Halving the interval in a table when first derivatives are given. Quart. J. Mech. Appl. Math. **6**, 128 (1953).

The formula $f(\frac{1}{2}h)=\frac{1}{2}\{f(0)+f(h)\}-\frac{1}{8}h\{f'(h)-f'(0)\}$ is used for midpoint interpolation. *W. E. Milne.*

Albrecht, J. Taylor-Entwicklungen und finite Ausdrücke für Δu und $\Delta\Delta u$. Z. Angew. Math. Mech. **33**, 41–48 (1953). (English, French and Russian summaries)

The author develops a number of finite difference expressions for Δu and $\Delta^2 u$ using square, triangular, and hexagonal plane grids, as well as three-dimensional grids.

H. Polachek (Carderock, Md.).

Mikeladze, Š. E. Approximate formulas for multiple integrals. Soobščeniya Akad. Nauk Gruzin. SSR **13**, 193–200 (1952). (Russian)

For expressing the integral

$$\iint_D f(x, y)dxdy = \sum_{\nu=0}^{r} A_\nu f(a_\nu, b_\nu) + R$$

in such a way that R vanishes when f is a polynomial of given degree, the author expands $f(x, y)$ and relates the A_ν to the integrals $\iint_D x^\alpha y^\beta dxdy$ in the obvious manner. Some special consideration is given to the cases when D is a rectangle and a circle. *A. S. Householder.*

Amble, Ole. A set of formulas for numerical integration. Norske Vid. Selsk. Forh., Trondheim **25** (1952), 38–41 (1953).

Formulas are presented for the numerical value of the integral of a function which is known at equidistant points in and adjacent to the interval of integration. For the formula of lowest order, a parabola is drawn through four consecutive points and an expression for the integral between the middle two of the four points obtained. By summing such expressions, it is shown that

$$\int_0^{nh} ydx = \frac{h}{24}(-y_{-1}+12y_0+25y_1+24y_2+24y_3\cdots +24y_{n-2}+25y_{n-1}+12y_n-y_{n+1}).$$

Similar formulas are indicated for fitting curves of higher degree. In an example the author shows that Simpson's formula is better than the above formula and worse than that of next higher degree. *S. Levy* (Washington, D. C.).

Myard, Francis. Résolution grapho-mécanique et simultanée des intégrales successives $\int cy^n xdy$ prises le long d'une courbe fermée. C. R. Acad. Sci. Paris **236**, 1947–1949 (1953).

Spencer, L. V. Calculation of peaked angular distributions from Legendre polynomial expansions and an application to the multiple scattering of charged particles. Physical Rev. (2) **90**, 146–150 (1953).

This paper discusses approximate methods of evaluating Fourier integrals which arise, in many physical problems, from the transformation of a slowly convergent series of spherical harmonics. The method makes as much use as possible of the qualitative information known about the integral and integrand. The latter is replaced by a sum of terms involving functions which have easily determined Fourier transforms, and which satisfy other criteria connected with the qualitative information known about integral and integrand, and with known values of the latter, or its integral or derivative, at chosen points in the range of integration. The evaluation of the function

$$F(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwx} \exp\left[-A\{1-\sigma^s K_s(\sigma)\}\right]d\sigma$$

is discussed in some detail, with illustrations of the appropriate approximations to use for large and small w. No theoretical indication is given of the expected accuracy corresponding to a given number of terms, though the calculated results compare well with those obtained by more exact methods.

The author also discusses the transformation of slowly convergent series of Legendre polynomials, obtaining, for example, the result

$$\sum_{l=0}^{\infty} g_l P_l(\cos\theta) = 2^{1/2}\int_\theta^{2\pi-\theta} d\phi(\cos\theta-\cos\phi)^{-\frac{1}{2}}\sum_{m=0}^{\infty}(-1)^m R(\phi+2\pi m)$$

where

$$R(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d(l+1)\sin\{(l+\tfrac{1}{2})r\}g(l+\tfrac{1}{2})$$

and $g(l+\frac{1}{2}) = g_l$ for integer l in $-\infty \leqq l+\frac{1}{2} \leqq \infty$, and is an unsymmetric function of $(l+\frac{1}{2})$. When the Legendre series converges slowly the integral series converges rapidly. This transformation and suggested method of integration are used finally to solve a typical scattering problem.

L. Fox (Teddington).

Davis, Philip. Errors of numerical approximation for analytic functions. J. Rational Mech. Anal. **2**, 303–313 (1953).

A general procedure is described for determining the error of approximation formulas in terms of Hilbert-type norms rather than in terms of the traditional derivatives. It is assumed that the function f to be approximated is a member of a space of functions all of which are analytic in a fixed region R, where R includes the interval of approximation. The particular case is discussed in detail where the functions are analytic and belong to H^2 in $|s| < 1$ and the approximation formulas (Simpson's rule, for instance) apply on the real interval $[0, 1/2]$. Let $E(f)$ be a linear functional expressing the error of the approximation formula. A norm $\|f\|$ is defined for the space and an integral representation for f is obtained. If one applies E to this representation and uses the Schwarz inequality, the relation $|f| \leqq \sigma\|f\|$ is obtained, where σ, the norm of the functional E, is independent of f. Values of σ are calculated for several important approximation formulas in the above special case. Methods are given for calculating $\|f\|$ in terms of the values of f at an infinite sequence of points having a limit point in R. Generalizations are described for the case where f is an analytic function of two or more complex variables and applications are made to approximation formulas of the Liebmann type for the Laplacian. *P. W. Ketchum* (Urbana, Ill.).

Collatz, Lothar. Über die Instabilität beim Verfahren der zentralen Differenzen für Differentialgleichungen zweiter Ordnung. Z. Angew. Math. Physik **4**, 153–154 (1953).

As has been shown by various writers the step-by-step numerical solution of an ordinary differential equation by means of central difference formulas (or equivalent formulas in terms of ordinates) may become unstable due, roughly speaking, to extraneous solution(s) introduced by the difference equation used to replace the differential equation. For the equation $y'' = f(x, y, y')$ the present paper shows the regions of the AB-plane (where $A = h^2 f_y$ and $B = hf_{y'}$) in which stability and instability occur. *W. E. Milne.*

Mitchell, A. R., and Craggs, J. W. Stability of difference relations in the solution of ordinary differential equations. Math. Tables and Other Aids to Computation **7**, 127–129 (1953).

In this note some well-known finite difference methods for integrating ordinary differential equations are analyzed with respect to numerical stability. *H. Polachek.*

***Klotter, K. Non-linear vibration problems treated by the averaging method of W. Ritz.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 125–131. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The author proposes what is usually known as Galerkin's method (but which, as he points out, should perhaps be attributed to Ritz) as a general method for obtaining approximations to the solutions $q(t)$ of the differential equation (*) $E(q, t)=0$; i.e., for suitably chosen $\psi_k(t)$ the approximation is given by $\bar{q}(t)=\sum_{k=1}^{n}a_k\psi_k(t)$ where the a_k are determined so that $\int_{t_0}^{t_1}E(\bar{q}, t)\psi_k(t)dt=0$, $k=1, 2, \cdots, n$. Using $\bar{q}=a_1\cos\Omega t+a_2\sin\Omega t$, $t_0=0$, $t_1=2\pi/\Omega$, the author illustrates how the method may be applied to obtain approximations to those solutions of (*) of period $2\pi/\Omega$ when

$$E(q, t)=a\ddot{q}+bg(\dot{q})+cf(q)-P\cos\Omega t.$$

A comparison with exact results is made in this instance when $g(\dot{q})=0$ and $f(q)=(\operatorname{sgn} q)^{n+1}q^n$, n an integer. No estimate of error is indicated for the general application of the method. *C. E. Langenhop* (Ames, Iowa).

Crandall, Stephen H. On a stability criterion for partial difference equations. J. Math. Physics **32**, 80–81 (1953).

In this note the author exhibits a partial difference equation which has an unstable solution but which has stable characteristics locally. He points out that an analysis for numerical stability based on the procedure discussed by O'Brien, Hyman and Kaplan [same J. **29**, 223–251 (1951); these Rev. **12**, 751] is not valid for this type of equation. *H. Polachek* (Carderock, Md.).

Lesky, Peter. Calcolo numerico: Ricerca di una funzione armonica soggetta a condizioni al contorno non lineari. Rend. Accad. Sci. Fis. Mat. Napoli (4) **19**, 5 pp. (1952) =Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo. no. **350** (1952).

A numerical method is given to solve the equation $\Delta u=0$ in a domain D where $\partial u/\partial n=a(Q, u)u+b(Q)$ for any point Q on the boundary of D. Here $a(Q, u)$ is assumed to be a polynomial in u with coefficients depending on Q. The kth approximation u_k is obtained by solving $\Delta u_k=0$ with the boundary condition $\partial u_k/\partial n=a(Q, u_{k-1})u_k+b(Q)$. The zeroth approximation u_0 is assumed to be constant. An illustrative example is given. Convergence questions are not discussed. *B. Friedman* (New York, N. Y.).

***Garwick, Jan V. On the numerical solution of integral equations.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 113–121. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

The early papers of Fredholm and Hilbert develop a theory of the integral equation $y(s)=\lambda\int_a^b K(s,t)\,y(t)dt+f(s)$ by substituting a finite sum for the integral. Also various quadrature formulae have been proposed for the numerical solution. These lead to a set of linear equations for the values of $y(s)$ at given pivotal points. The author proposes the quadrature formula of Gauss, which has already been applied by Nyström and others. Another method of numerical solution is to substitute for $y(s)$ a linear combination of linearly independent functions. This results in a set of linear equations for the combination coefficients. The choice of polynomials was proposed by Lösch in an earlier paper. The author chooses Legendre polynomials. It seems that these polynomials are useful for equations with kernels $K(s,t)=A(s,t)h(s,t)+B(s,t)$ with either $h(s,t)=\ln|s-g(t)|$ or $h(s,t)=|s-g(t)|^{\alpha-1}$, $\alpha>0$, with A and B as regular functions in the neighborhood of $s=g(t)$. The paper concludes with recursion formulae for the integrals

$$\int_{-1}^{1}h(s,t)P_n(t)dt=H_n(s),$$

where P_n denotes the Legendre polynomial of degree n. There are no estimates of errors and no numerical examples. *H. Bückner* (Holloman, N. M.).

Fox, L., and Goodwin, E. T. The numerical solution of non-singular linear integral equations. Philos. Trans. Roy. Soc. London. Ser. A. **245**, 501–534 (1953).

Integral equations of second and first kind with "well-behaved" kernels of both the Fredholm and the Volterra type are considered. For Fredholm equations of the first kind some brief remarks are made so as to show that a counterpart to the nonexistence of solutions exists in the numerical field. Volterra equations of the first kind are transformed to those of the second kind by differentiation (already proposed by Richardson and Gaunt). Thus the numerical methods refer mainly to equations of the second kind $y(s)=\lambda\int_a^b K(s,t)y(t)dt+f(s)$ or $y(s)=\int_a^s K(s,t)y(t)dt+f(s)$. It is a well known method to substitute quadrature formulae for the integral. This leads to a set of ordinary linear equations for approximate values of $y(s)$ at the pivotal points of the formulae. Also eigenvalues and eigenfunctions can be calculated this way. So far, earlier papers (of Fredholm, Hilbert, Hohenemser, Nyström, Prager, and various other authors) have dealt with Gauss-formulae and with equidistant pivotal points formulae. The authors also use a weighted sum formula, namely, $\int_a^{a+nh}g\,dt=\sum_{i=0}^{n}p_ig_i$ with $g_i=g(a+ih)$. Instead, using the formula in direct computation, they set $\sum p_ig_i=\frac{1}{2}g_0+g_1+\cdots+g_{n-1}+\frac{1}{2}g_n+\Delta$, where Δ is a correction term consisting of differences of the g_k. Then the corresponding system of linear equations for the approximate pivotal values of $y(s)$ is solved by approximation of increasing order with Δ calculated from the preceding approximation, the first approximation being obtained with Δ neglected. They prefer relaxation methods for the Fredholm type with the Δ-term as unliquidated residual. The method is modified for the Volterra type.

There are also some remarks (which the reviewer believes are not new) on iteration, on the deferred approach to the limits and on the application of the Laplace transformation to Volterra equations with difference-kernels. All demonstrations are on the basis of some numerical examples. The list of references is mainly restricted to British papers. *H. Bückner* (Holloman, N. M.).

Mineur, Henri, et Peyturaux, Roger. Exploitation numérique de l'équation intégrale d'assombrissement au bord solaire pour obtenir la fonction source. Ann. Astrophysique **15**, 383–401 (1952).

In an analysis of the observations relating to the continuous spectrum of the sun, the following problem is a fundamental one: Given that a function $\phi(\mu)$ $(0\leq\mu\leq1)$ is determined in terms of a "source function" $S(\tau)$ in such a

way that

$$\phi(\mu)=\int_0^\infty S(\tau)e^{-\tau/\mu}d\tau/\mu,$$

to determine $S(\tau)$ from the values of $\phi(\mu)$, known observationally, at a certain number of discrete points $\mu_1, \cdots, \mu_n$ (say). (In practice the number of ordinates at which $\phi(\mu)$ is known is about ten.) By considering a given practical problem the authors illustrate the known fact that any numerical method based on interpolation and which expresses $\phi(\mu_i)$ in terms of the values of $S(\tau)$ at certain suitably chosen "representative" points τ_j is entirely unreliable. Methods based on assuming for $S(\tau)$ analytical expressions involving a parameter α and a certain number of constants a, b, etc., on which $\phi(\mu)$ depends linearly are next investigated. It appears that in general there is considerable arbitrariness in the forms for $S(\tau)$ which may be chosen and in the values of the constants α, a, b, etc., which are determined for any particular chosen form. However, from a detailed consideration of a numerical example along these latter lines the authors suggest that an empirical determination of $S(\tau)$ is possible upon which one may place fair reliance.

S. Chandrasekhar (Williams Bay, Wis.).

***Young, David M. The use of conformal mapping to determine flows with free streamlines.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 125–136. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

Numerical and analytic methods are given for the calculation of the parameters occurring in the Schwarz-Christoffel transformations as used in the study of flows with free streamlines. *C. Saltzer* (Cleveland, Ohio).

***Garrick, I. E. Conformal mapping in aerodynamics, with emphasis on the method of successive conjugates.** Construction and applications of conformal maps. Proceedings of a sympsoium, pp. 137–147. National Bureau of Standards, Appl. Math. Ser. No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

The derivation of the Theodorsen-Garrick method is reviewed and a method for the approximation of conjugate functions is discussed. Applications of the methods discussed to integral equations with singular kernels, the incompressible flow past bodies of revolution, Poisson's equation, biplane wing theory, cascades of airfoils, design of airfoil profiles and velocity correction formulas are indicated. [The discussion of the approximation of conjugate functions in this paper overlaps the treatment by the reviewer in his thesis [Brown Univ., 1949].] *C. Saltzer.*

***Ostrowski, A. M. On the convergence of Theodorsen's and Garrick's method of conformal mapping.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 149–163. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

The methods of Warschawski [Quart. Appl. Math. **3**, 12–28 (1945); these Rev. **6**, 207] are used to generalize and extend his results to the case where the non-homogeneous term of the Theodorsen-Garrick integral equation is replaced by a more general function which improves the convergence of the iteration procedure used in approximating the solution. In addition, sufficient conditions are given for the convergence of the approximations of derivatives of a given order. *C. Saltzer* (Cleveland, Ohio).

***Ostrowski, A. M. On a discontinuous analogue of Theodorsen's and Garrick's method.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 165–174. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

In the solution by iteration of the generalized Theodorsen-Garrick integral equation, since the integrals cannot be evaluated in closed form, the integrals are replaced by sums and the integral equation is replaced by a system of simultaneous non-linear equations. The theorems on the convergence of the iteration procedure for the integral equation are not applicable to the system of simultaneous equations. It is proved under the same hypotheses as are required in the case of the integral equation that the iterative process converges. In addition, it is shown that the bound on ϵ is the least possible. [With the exception of the result concerning ϵ, a similar discussion was given by the reviewer for the integral equation treated by Warschawski, i.e., the case where $\gamma(\phi)=\phi$, in his thesis [Brown Univ., 1949; see also Bull. Amer. Math. Soc. **56**, 177 (1950)]. In addition, the estimates given by the reviewer are uniform with respect to the order of the polynomial of interpolation.]

C. Saltzer (Cleveland, Ohio).

***Swinford, Lee H. An approximate method for conformal mapping.** Construction and applications of conformal maps. Proceedings of a symposium, p. 225. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

A modification of the Koebe-Carathéodory method for the conformal mapping of a simply-connected region is given which is suitable for manual computation and it is remarked that while the convergence is of the same order of magnitude as the estimate given by Ostrowski [Jber. Deutsch. Math. Verein. **38**, 168–182 (1929)] for the original procedure, the estimate of the bound for the constant for this method is smaller. *C. Saltzer* (Cleveland, Ohio).

Gray, H. J., Jr., Rubinoff, M., and Sohon, H. A technique for real time simulation of a rigid body problem. Math. Tables and Other Aids to Computation **7**, 73–77 (1953).

Gregg, C. V. Reciprocal nomograms. Math. Gaz. **37**, 90–95 (1953).

Schmid, H. Eine graphische Methode des mehrfachen Einschneidens mit Hilfe der Tienstra'schen Symbolik. Abh. Dokumentationszentrums Technik Wirtschaft no. **8**, 10 pp. (1952).

Dubois, Fr. Nouveaux appareils pour opérations mathématiques. Génie Civil **130**, 123–125, 150–153, 169–171 (1953).

A new digital computor. Electronic Engrg. **25**, 201 (1953).

Pateman, J. E. The application of sampling methods to analogue integrators. Elliott J. **1**, 110–112 (1953).

Ostapenko, V. N. On electromodelling of problems of filtration from canals of trapezoidal section. Ukrain. Mat. Žurnal **4**, 97–99 (1952). (Russian)

Nedelkow, I. Bestimmung der Schwerkraftsverteilung in der Nähe homogener Körper mittels elektrischer Modelle. Izvestiya Bulgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. 1 (1950), 263–271 (1951). (Bulgarian. Russian and German summaries)

Lah, Ivo. Die Taylorsche Reihe der generalisierten Poukkaschen Funktion und ihre Anwendung. Mitt. Verein. Schweiz. Versich.-Math. 53, 78–91 (1953).

Poukka's formula is used in the approximation of life annuity values at a rate of interest i' by means of basic tables calculated at a rate i [Skand. Aktuarietidskr. 6, 137–152 (1923)]. It has been generalized by the author [same Mitt. 51, 191–210 (1951); these Rev. 13, 843] in terms of a function involving three successive orders of repeated sums of $D_x=v^x l_x$. This function is insensitive to changes in i and has been ascribed a constant value in the approximations. But here the author uses two and three-termed Taylor series developments to derive new and accurate approximations for life annuity values at rate i'.

H. L. Seal (New York, N. Y.).

ASTRONOMY

Mendes, Marcel. Application des transformations canoniques à la résolution du problème des deux corps. Bull. Astr. 16, 321–325 (1952).

The author illustrates with the help of the two-body problem his method of integrating Hamiltonian systems. This method is only a slight generalization of the well-known method of Jacobi. The author admits that this method of solving the two-body problem cannot be compared in simplicity with the classical method.

D. C. Lewis, Jr. (Baltimore, Md.).

Siegel, Carl Ludwig. Über eine periodische Lösung im ebenen Dreikörperproblem. Math. Nachr. 4, 28–35 (1951).

The author obtains a two-parameter family of periodic solutions of the planar three-body problem. The coordinates are expressed as power series in two variables, each of which is a sinusoidal function of time, and convergence is established with relative ease. For extreme values of the parameters one obtains known solutions of the restricted problem and of the lunar problem of Hill. *W. Kaplan.*

Lemaître, G. Coordonnées symétriques dans le problème des trois corps. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 38, 582–592 (1952).

By using a clever transformation the author shows that the three-body problem in the plane for the case of equal masses is equivalent to the motion of a point in three-dimensional space under the combined action of a scalar potential function and a vector potential function. The general case is also treated by this same transformation but the results are more difficult to interpret.

D. C. Lewis, Jr. (Baltimore, Md.).

Lemaître, G. Coordonnées symétriques dans le problème des trois corps. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 38, 1218–1234 (1952).

Using the coordinates introduced in an earlier paper [see the preceding review], the author shows how the regularization of double collision may be removed by means of a conformal transformation and he gives the explicit expression for the Hamiltonian expressed in terms of the new variables.

D. C. Lewis, Jr. (Baltimore, Md.).

Littlewood, J. E. On the problem of n bodies. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 143–151 (1952).

Let the n bodies have masses $m_i \geqq 0$, with $m_n > 0$, and assume the center of mass fixed, so that the phase space M is of dimension $6n-6$. Let E denote a measurable subset of M; let $P(t)$ denote the position at time t of the point at P when $t=0$. Three theorems are proved. 1. If E is such that each solution $P(t)$, with P in E, has the property that for $t \geqq 0$ each body is confined to a spherical annulus in 3-space and no two annuli intersect, then for almost all P in E the same is true for $t \leqq 0$. 2. Let E have the property that each solution $P(t)$ for P in E leads to no collision for $t \geqq 0$ and for $t \geqq 0$ has a bounded diameter in 3-space; then for almost all P of E the same is true for $t \leqq 0$. 3. For almost all P of a set E either the conditions in 2 hold for $t \geqq 0$ or else, for every $p>0$, $P(mp)$ is in E for infinitely many positive integers m. *W. Kaplan* (Zurich).

***Sokolov, Yu. D. Osobye traektorii systemy svobodnyh material'nyh toček. [Singular trajectories of a system of free material points.]** Monografii Instituta Matematiki, vyp. I. Akad. Nauk Ukrainskoĭ SSR, Kiev, 1951. 126 pp. 6 rubles.

For the qualitative treatment of dynamical questions investigation of singular points and singular trajectories of equations of dynamical origin is of the highest importance. The author considers the singular trajectories of a system of $n(\geqq 3)$ particles P_i of masses m_i $(i=1, 2, \cdots, n)$, which attract or repel each other, the interaction between P_i and P_j $(i \neq j)$ having magnitude $m_i m_j |f(r_{ij})|$ and representing an attraction or repulsion according as f is negative or positive. It is assumed that $f(r)=dF(r)/dr$ is analytic for positive r and may have singularities at the points $r=0$ and $r=\infty$ on the real axis.

The monograph consists of four chapters. Chapter I contains some general remarks concerning the regular motion of the system and the singularities of the integrals of motion. In §1 a lower bound for the radii of convergence of the expansions of the coordinates of the particles about an instant in the regular motion is determined. In the particular case of three bodies under Newtonian attraction the value of this lower bound obtained is approximately one and a half times greater than the value, given by the corresponding formulas of Sundman [Acta Math. 36, 105–179 (1912)] and Mendes [Ann. Fac. Sci. Univ. Toulouse (3) 27, 1–169 (1935)]. In §§2–3 certain statements of Weierstrass [see Mittag-Leffler, Acta Math. 35, 29–65 (1911)] and Poincaré [Math. Pures Appl. (4) 2, 151–217 (1886)] on the analytical representation of functions characterizing regular motion of the system, and of Painlevé [Leçons sur la théorie analytique des équations différentielles, Hermann, Paris, 1897; lecture 22] on the behavior of the system in a neighborhood of a singular point $t=t_1$, are generalized and extended. In §4 on the basis of the generalized Lagrange-Jacobi equality

$$\frac{1}{2}\frac{d^2 J^2}{dt^2} = \sum m_i m_j F_1(r_{ij}) + 2h, \tag{1}$$

where $F_1(r)=2F(r)+rf(r)$, $J^2=M^{-1}\sum m_i m_j r_{ij}^2$ is the moment of inertia of the system about its center of mass and M is the total mass, sufficient conditions to the effect that J^2 tends toward a finite value or increases indefinitely as $t\to t_1$ are given. If the motion remains regular in the whole interval from $t=0$ to $t=+\infty$, then from (1) some simple conclusions as to the behavior of J^2 as $t\to\infty$ and, consequently, on the stability of the system under consideration can be made. In this connection the author draws attention to an erroneous conjecture made by Jacobi [Vorlesungen über Dynamik, lecture 4; Supplementband of his Werke, Reimer, Berlin, 1884], which has been later repeated by Birkhoff [Dynamical systems, Amer. Math. Soc. Colloq. Publ., vol. 9, New York, 1927, Chapt. 9] and Chandrasekhar [Principles of stellar dynamics, Univ. of Chicago Press, 1942, chapt. 5; these Rev. 4, 57]. From the relation $\lim_{t\to\infty} J^2=\infty$ the above mentioned authors draw the conclusion that at least one of the bodies would then recede infinitely far from their common center of mass, while, in fact, one may only infer that $\lim_{t\to\infty} r\to\infty$, where r is the greatest of the mutual distances between the particles.

In chapter II trajectories of double collision

$$(\lim_{t\to t_1} J^2=J_1^2>0)$$

in the generalized three bodies problem are considered, assuming that

$$\lim_{r\to+0} r^{2\alpha+1}f(r)=-2\alpha<0. \tag{2}$$

Ten cases of integrability of the equations of spatial, planar and collinear motion are obtained, some of which are generalizations of previously known results and some are new. Analytic weapons sufficiently powerful to deal with the singularity of double collision were first developed by Sundman [loc. cit.] and Levi-Cività [Acta Math. 42, 99–144 (1920)]. The method, used by the author in §§3–4 and 6, is essentially that of Levi-Cività, suitably generalized. In §6 two invariant relations are established, characterizing trajectories of double collision, and partial differential equations are given which are satisfied by the two functions which occur in the conditions for double collision. The case $F(r)=1/r^{2\alpha}$ is investigated in more detail; for $\alpha=1/2$ the corresponding formulas of Levi-Cività [Ann. Mat. Pura Appl. (3) 9, 1–32 (1903)] and Bisconcini [Acta Math. 30, 49–92 (1905)] are obtained, and certain errors in the coefficients obtained by Bisconcini are noted and corrected.

In §7 a second and more effective method (based on the reduction of the equations of motion to some special forms) which permits a more uniform treatment of all cases (for $\alpha=1$ the first method ceases to be applicable) is given. In §8 some particular and limiting cases of motion are considered such as motions, having a plane or an axis of symmetry, planar motions, and the limiting "restricted problem of three bodies" treated by Hill [Amer. J. Math. 1, 5–26, 129–147, 245–260 (1878)=Coll. Math. Works, Carnegie Inst. of Washington, vol. 1, pp. 284–335, 1905].

Chapter III is concerned with trajectories of general collision ($\lim_{t\to t_1} J^2=0$) under the assumption (2). Generalizing a conjecture of Sludskiĭ [Mat. Sbornik 9, 536–545 (1879)] and Weierstrass [loc. cit., p. 58] which states that for a general collision it is necessary that the angular momentum of the particles be zero about every axis through their common center of mass, and which was first proved by Sundman [Acta Soc. Sci. Fennicae 34, no. 6 (1907)] and Chazy [Bull. Astr. 35, 321–389 (1918)], the author proves in §1 the following theorems. Theorem 1. If for a system of n particles, moving in a p-dimensional Euclidean space, $\lim_{t\to t_1} J^2=0$, and in the case $\lim_{t\to t_1} U=+\infty$, in the neighborhood of $r_{ij}=0$ the inequality

$$(2-B)U+\sum r_{ij}\frac{\partial U}{\partial r_{ij}}\geqq 0$$

holds, where $-U$, the potential energy of the system, depends only on the mutual distances of the particles, and B is a positive constant, then in the motion of the system referred to the center of mass all the constants of the angular momentum of the particles are zero. Theorem 2. If all the constants of the angular momentum of a system of n particles in their motion in a p-dimensional space ($n\leqq p$) referred to their common center of mass are zero, the motion is in a fixed hyperplane of $(n-1)$-dimensions through their center of mass. From theorem 1 for $p=3$, $U=\sum m_i m_j/r_{ij}$, $F(r)=1/r$ the above mentioned conjecture of Sludskiĭ and Weierstrass follows, and from theorem 2 for $p=3$, $n=3$, $U=\sum m_i m_j/r_{ij}$, $F(r)=1/r$ a well-known theorem of Dziobek [Die mathematischen Theorien der Planeten-Bewegungen, Barth, Leipzig, 1888] follows also.

In §§2–4 for the case $n=3$ a generalized method of Sundman is used to investigate the motion in the neighborhood of the instant of triple collision, assuming that for small r ($\alpha\neq 1$) the inequality

$$(1-\alpha)\frac{d}{dr}[r^{2\alpha}F(r)]\geqq 0$$

holds. Further, it is proved that as t tends toward the instant of triple collision and $\alpha\neq 1$, the three particles tend to form one of the two limiting configurations: an equilateral triangle or a rectilinear configuration, as obtained by Sundman in the case of Newtonian attraction. In §6, after the equations of motion in the plane have been reduced to four first-order differential equations with J as the independent variable, some results of Bohl [Bull. Soc. Math. France 38, 5–138 (1910)], Cotton [Ann. Sci. Ecole Norm. Sup. (3) 28, 473–521 (1911)] and others on asymptotic solutions of differential equations are applied to establish the existence of trajectories of collision. In §§7–9 analytical representations of these trajectories around $J=0$ are given for the two previously mentioned limiting configurations in the case of planar and spatial motion. §10 deals with motions symmetric in space.

Chapter IV is concerned with the case when the particles recede indefinitely far from each other ($\lim_{t\to t_1} J^2=+\infty$). It is assumed that $f(r)$ is analytic for positive r, continuous for $r=0$, and increases indefinitely as $r\to+\infty$ in such a way that

$$\lim_{r\to\infty} r^{1-2\beta}f(r)=2\beta>0.$$

In §1 it is shown that of necessity $\beta>1$, and three possible cases are considered: (i) $\lim_{t\to t_1} \underline{p}=0$ ($\underline{p}=\underline{r}/J$), (ii) $\inf \underline{p}$ is positive in the neighborhood of $t=t_1$, (iii) $\underline{p}$ oscillates between zero and some finite upper bound ($\underline{r}$ denotes the smallest of the three distances r_0, r_1, r_2 at the instant t). In case (i) the ratios $r_i/J=p_i$ ($i=0, 1, 2$) as well as $V=J^{-2\beta}U=J^{-2\beta}\sum m_i m_j F(r_{ij})$ ($i\neq j$) tend to finite limits. Making appropriate assumptions, it is shown in §2 that in all three cases $R^2=J^{-2\beta}(dJ/dt)^2$ tends toward a finite positive limit R_1^2 as $t\to t_1$. Under the assumption $\lim_{t\to t_1} r_i=+\infty$, which is always satisfied in case (ii), it is shown in §3 that $\lim_{t\to t_1} 2V=R_1^2$. Considering planar motion, it is shown that in the cases (i) and (ii) all the p_i tend toward finite limits

as $t \to t_1$, and that one of the possible three configurations is again the limiting equilateral triangle. In §4 the existence of planar and spatial trajectories, and the existence of limiting configurations corresponding to the previously mentioned three cases, is proved. In §5 trajectories in the case of spatial motion, having a plane or an axis of symmetry, are considered, the latter case being investigated in detail. Finally in §6 the trajectories of the asteroid, when the latter recedes indefinitely, in the spatial restricted problem of three bodies are studied. This case has some peculiarities, and the results obtained cannot be derived from the general problem by passage to the limit, in contrast to the case for the trajectories of double collision.

With the publication of this monograph a certain period of work (1934–1951) of the author on the problem stated in the beginning may be considered as finished.

E. Leimanis (Vancouver, B. C.).

***Bondi, H. Cosmology.** Cambridge Monographs on Physics. Cambridge at the University Press, New York, N. Y., 1952. vi+179 pp. $4.50.

The author's aim is "to present cosmology as a branch of physics in its own right"; in this he succeeds very well indeed. The book deals quite competently, and for the most part adequately, with the major cosmological theories since that based upon Einstein's 1917 modification of the field equations of general relativity. Bondi's emphasis on cosmology as an independent science is perhaps at least in part due to his desire to build up the deductive, or a prioristic, approach as opposed to the extrapolative method which has characterized the great bulk of physical science. In the theory of relativity, as in classical mechanics, the laws of motion and the field equations, inductively derived from terrestial or solar-system experience, are extended from the realm of more immediate experience to the entire cosmos, on the assumption that the "constants" (such as that of gravitation) derived here and now are in fact constant throughout the whole of the space-time universe. In the deductive approach, on the other hand, assumptions concerning the nature of the world at large are laid down a priori, and physical law deduced therefrom as consequence.

In either case, the leitmotiv of the problem here considered is to be found in the "Cosmological Principle", the assumption that for properly chosen real or ideal observers the over-all views of the world are indistinguishable. For the extrapolative theories, such as the theory of relativity, this "principle" is merely an aid to the formulation of a problem which is to be attacked within the framework of the extrapolated physical laws; for the deductive theories, such as Milne's kinematical relativity, it is an a priori requirement, a sort of categorical imperative, to which physical experience must conform. Of special note is the case of the "Perfect Cosmological Principle", which would require that the world-views obtained by equivalent observers are in addition stationary (but not necessarily static); upon it are based the Bondi-Gold steady-state theory (deductive) and the Hoyle theory of continuous creation (mainly extrapolative).

Bondi's book is devoted, as announced in his introductory Part I, to an exposition and application to the observed universe of these two types of cosmological theories. The relevant observational evidence is marshalled in Part II; unfortunately, the manuscript was prepared before Baade's announcement of the necessity of materially increasing the distance scale for extragalactic objects. This proposed change of scale would considerably ameliorate the dysphoria into which the extrapolative theories have fallen because of the painfully short time-scale implied by the interpretation of the nebular redshifts as a Doppler effect.

In Part III the various theories are developed and examined in the light of the observations. The reasons given for the rejection of many of them should be critically scrutinized, for they are based at least in part upon the admittedly too-short time-scale derived from the previously accepted distance criteria and by the author's predilection for a steady-state theory. Jordan's theory, in which the "constant" of gravitation varies with cosmological time, is given short shrift, mainly because of Jordan's assumption that the new matter enters the universe in concentrated form—and not, as in the author's steady-state theory, more or less uniformly throughout space.

The book ends with a stimulating chapter on the present position in cosmology, suggesting experimenta crucis which may lead to a decision between the various cosmological theories which have been proposed. *H. P. Robertson.*

Levee, Richard D. A gravitationally contracting stellar model. Astrophys. J. **117**, 200–210 (1953).

The author integrates numerically the equations of radiative equilibrium for a gravitationally contracting model neglecting radiation pressure. The resulting distributions are close to those of the standard model and give rise to a time scale of 5×10^7 years for the sun.

R. G. Langebartel (Urbana, Ill.).

Bandyopadhyay, G. A note on the limiting mass of a rotating white dwarf. Bull. Calcutta Math. Soc. **44**, 89–91 (1952).

The author sets up the second-order differential equation governing the internal structure of degenerate-gas configurations (taking account of relativistic corrections) rotating like a rigid body, and proceeds to construct an approximate solution of his partial differential equation by the method of Milne [Monthly Not. Roy. Astr. Soc. **83**, 118–147 (1923)] and von Zeipel [ibid. **84**, 665–683 (1924)] on the assumption that the rotation is so slow as to render the squares and higher powers of the centrifugal-force term negligible. He shows that, within the scheme of this approximation, the limiting value of the mass of a relativistically degenerate configuration, established previously by Chandrasekhar for non-rotating configurations, continues to hold good unaltered—in contrast to another recent result by Hoyle [ibid. **107**, 231–236 (1947)], who on the assumption of a special type of differential rotation concluded that there was no limit to the mass which a so rotating degenerate configuration may attain. *Z. Kopal* (Manchester).

Ambarzumjan, W. A. Zur Theorie der Fluktuationen in der scheinbaren Verteilung der Sterne an der Sphäre. Abh. Sowjet. Astron. Folge II=Sowjetwissenschaft, Beiheft **27**, 155–195 (1951).

This monograph presents a coherent account of the work of Ambarzumjan and his pupils on the problems in stochastic theory which arise in stellar statistics from the non-uniform distribution of the interstellar absorbing matter. The basic assumption which is made regarding the latter is that the absorbing matter occurs in the form of discrete clouds and that the number of clouds $n(s)$ which occur in a distance s is governed by a Poisson distribution with a variance νs where ν is a given constant.

The treatment is divided into 13 sections. Sections 1 and 2 are introductory. Sections 3 and 4 treat the problem of the

fluctuations in the counts of extragalactic nebulae due to the cloud structure of the absorbing matter in the galaxy idealized as a plane-parallel slab. The treatment is somewhat more general than in the earlier publication of the author [Bull. Abastumani Observatory 4, 17–23 (1940)]. In particular, the author considers the probability $u_s(\tau)$ that in a line of sight extending to a distance s from the origin, the total optical thickness of the intervening clouds is less than τ. If ξ is the distance measure in the unit νs, the author shows that $u_\xi(\tau)$ is governed by the integral equation

$$\frac{du_\xi(\tau)}{d\tau}=-u+\int_0^\tau u_\xi(\tau-\sigma)dF(\sigma) \tag{1}$$

where $F(\sigma)$ is related to the probability distribution of clouds of different transparency factors $q(=e^{-\sigma})$ by $\phi(q)=1-F(\sigma)$. Letting

$$Q_k=\int_0^\infty e^{-k\tau}du_\xi(\tau),$$

the author shows by using equation (1) that

$$Q_k=\exp\{-\nu s(1-q_k)\},\quad q_k=\int_0^\infty e^{-k\sigma}dF(\sigma).$$

Sections 5, 6, and 7 deal with the problem of the fluctuations in brightness, u, of the Milky Way. The author derives the integral equation

$$g(u)+\frac{dg}{du}=\int_0^1\psi(q)g\left(\frac{u}{q}\right)\frac{dq}{q}, \tag{2}$$

governing the frequency distribution $g(u)$ of u. In (2), $\psi(q)$ is the frequency of occurrence of a cloud with a transparency factor q. The solution of (2) for the case when $\psi(q)$ is a delta function is given. [The general solution of (1) without any restriction on $\psi(q)$ has been found by S. Chandrasekhar and G. Münch, Astrophys. J. 115, 94–102 (1952); these Rev. 13, 786. Also the more general integral equation

$$g(u,\xi)+\frac{\partial g}{\partial u}+\frac{\partial g}{\partial \xi}=\int_{u/\xi}^1\psi(q)g\left(\frac{u}{q},\zeta\right)\frac{du}{q} \tag{3}$$

which governs the case when the system considered is of finite extent has also been treated by Chandrasekhar and Münch, ibid. 112, 380–392 (1950); 114, 110–122 (1951); these Rev. 12, 644; 13, 249.] Section 8 deals with the problem which has also been treated under more general conditions by Chandrasekhar and Münch [ibid. 113, 150–165 (1951); these Rev. 12, 644].

Sections 9–12 break new ground by including in the description the fluctuations in star density in addition to the fluctuations in the interstellar absorption. If one supposes that the system one considers is so far away that one may neglect the decrease in the brightness of the stars with increasing distance, then the integral equation governing the distribution of the observed intensity is

$$\Psi(I)=(1-\lambda)\int_0^1\Psi(I/q)\psi(q)dq+\lambda\int_0^I\Psi(I-i)dB(i), \tag{4}$$

where $B(i)$ is the probability that a star has a brightness less than i and $\lambda=\nu/(\nu+n_*S)$ where ν is the average number clouds per unit length in the line of sight, n_* is the number of stars per unit volume, and S is the area of the sky viewed. The author shows that by using equation (4) the moments of I can be evaluated. Thus the dispersion is given by

$$\frac{\overline{(I-\bar I)^2}}{\bar I^2}=\frac{1}{\lambda}\frac{\overline{(1-q)^2}}{1-\bar q^2}+\frac{1-\lambda}{\lambda}\frac{\bar q^2-\bar q^2}{1-\bar q^2}+\frac{1-\lambda}{\lambda}\frac{\bar i^2-\bar i^2}{\bar i^2}\frac{(1-\bar q)^2}{1-\bar q^2}.$$

Attempts to generalize this simple problem are not altogether successful. An attempt is also made in these last sections to obtain the angular correlation of the fluctuations in brightness of the Milky Way; again the results are not conclusive [see Chandrasekhar and Münch, ibid. 115, 103–123 (1952); these Rev. 13, 786]. *S. Chandrasekhar.*

RELATIVITY

Jánossy, L. Über die physikalische Interpretation der Lorentz-Transformation. Ann. Physik (6) 11, 293–322 (1953).

German version of the author's paper in Acta Phys. Acad. Sci. Hungar. 1, 391–422 (1952); these Rev. 14, 506. *H. P. Robertson* (Pasadena, Calif.).

Ives, Herbert E. The Fitzgerald contraction. Sci. Proc. Roy. Dublin Soc. (N.S.) 26, 9–26 (1 plate) (1952).

Kohler, Max. Die Formulierung der Erhaltungssätze der Energie und des Impulses in der allgemeinen Relativitätstheorie. II. Z. Physik 134, 286–305 (1953).

Kohler, Max. Zur Herleitung der Feldgleichungen in der allgemein-relativistischen Gravitationstheorie. Z. Physik 134, 306–316 (1953).

In an earlier paper [Z. Physik 131, 571–602 (1952); these Rev. 14, 416] the author proposed a theory of gravitation based on the geometry of a 4-space with two metric tensors, and constructed an energy momentum tensor in the case of weak gravitational fields. In the first paper above, the author studies the energy-momentum tensor for strong gravitational fields. In the second paper he studies the field equations resulting from a more general variational principle. *A. Schild* (Pittsburgh, Pa.).

O'Brien, Stephen, and Synge, John L. Jump conditions at discontinuities in general relativity. Communications Dublin Inst. Advanced Studies. Ser. A. no. 9, 20 pp. (1952).

Les auteurs se préoccupent d'énoncer de manière précise les conditions relatives aux discontinuités du tenseur métrique $g_{\lambda\mu}$, de ses dérivées et du tenseur d'impulsion-énergie $T_{\lambda\mu}$, à la traversée d'une hypersurface S, en théorie relativiste de la gravitation. De telles considérations sont, de l'avis du rapporteur, fondamentales pour l'axiomatique de la théorie. Le même problème, avec des résultats analogues, a été traité par le rapporteur [A. Lichnerowicz, Problèmes globaux en mécanique relativiste, Hermann, Paris, 1939; ces Rev. 1, 282] en liaison avec le problème de Cauchy, sans que les auteurs semblent en avoir eu connaissance. Les auteurs emploient un intéressant procédé de passage à la limite en introduisant un tenseur métrique dépendant d'un paramètre ϵ et en interprétant les résultats à l'aide d'un espace à 5 dimensions. Ce procédé repose sur un certain nombre d'hypothèses nécessairement un peu arbitraires et conduit aux résultats suivants: si $x^0=0$ représente localement l'hypersurface S (supposée n'être pas tangente au cône élémentaire) les quantités $g_{\lambda\mu}$, $\partial_0 g_{ij}$ $(i,j=1,2,3)$, $T_\lambda{}^0$, sont continues à la traversée de S. Le résultat relatif aux

T_λ^μ coincide avec celui du rapporteur et constitue la base véritable des "équations du mouvement". Un changement de coordonnées une fois continûment différentiable, conservant S point par point et les différentes données peut faire apparaître ou disparaître des discontinuités éventuelles des $\partial_\alpha g_{\mu\nu}$ et il ne semble pas intéressant au rapporteur d'introduire de telles discontinuités [voir A. Lichnerowicz, Etude mathématique de theories relativistes de la gravitation et de l'électromagnétisme, t. I, cours ronéotypé du Collège de France, Paris, 1953]. *A. Lichnerowicz.*

Horváth, J. I., und Moór, A. Entwicklung einer einheitlichen Feldtheorie begründet auf die Finslersche Geometrie. Z. Physik **131**, 544–570 (1952).

Les auteurs, conformément au travail de l'un d'eux [Horváth, Physical Rev. **80**, 901 (1950)], se proposent de construire une théorie "unitaire" de la gravitation et de l'électromagnétisme, basée sur la géométrie finslerienne telle qu'elle a été développée par E. Cartan et par Varga [Publ. Math. Debrecen **1**, 7–17 (1949); ces Rev. **11**, 134]. Les deux premières sections sont consacrées à un court exposé de la géométrie finslerienne. Les auteurs utilisent d'une manière essentielle la notion, due à Varga, d'espace riemannien osculateur à un espace de Finsler le long d une famille F, à 1 paramètre, d'éléments linéaires. Ils sont ainsi amenés à introduire, au lieu du tenseur de courbure de Cartan $R_k^i{}_{rs}$, le tenseur principal de courbure de Varga $H_k^i{}_{rs}$, qui jouit de la propriété de coïncider le long de F avec le tenseur de courbure riemannienne $R_k^i{}_{rs}$ d'un espace riemannien osculateur, et qui est lié au tenseur de Cartan par la relation

$$H_k^i{}_{rs}=R_k^i{}_{rs}-A_k^i{}_p R_\phi^p{}_{rs}$$

où A désigne le tenseur de torsion et où $R_\phi^p{}_{rs}=R_k^p{}_{rs}l^k$ (l^k vecteur unitaire). Comme généralisation du tenseur de Ricci, ils adoptent ainsi le tenseur $R_{ik}=H_i^r{}_{kr}$ ($R=g^{ik}R_{ik}$). Cela posé, la métrique finslerienne introduite est supposée de type hyperbolique normal pour tout élément linéaire et les deux dernières sections sont consacrées à la construction des équations du champ. A l'aide d'un principe variationnel, les auteurs sont conduits, dans le cas du vide matériel, à un premier groupe d'équations

$$(*) \qquad R_{ik}-\tfrac{1}{2}g_{ik}R=0$$

complétées par

$$R_\phi^p{}_{rs}=0,$$

ce qui s'interprète géométriquement par l'existence d'un parallélisme absolu des éléments linéaires et conduit à l'existence, pour les premiers membres de (*), des conditions de conservation classiques. Le second groupe d'équations, qui porte sur un champ électromagnétique F_{ij}, est introduit de manière très arbitraire; S_{ij} étant le tenseur d'impulsion-energie de ce champ, les auteurs posent

$$A_{ik}=-\tfrac{1}{2}(\nabla_h S_{ik}+\nabla_i S_{hk}+\nabla_k S_{hi}).$$

Cette théorie ne paraît pas physiquement satisfaisante, le cas $A_{ik}=0$ soulevant, semble-t-il, des difficultés particulières de cohérence interne et elle n'est guère unitaire par suite de l'hétérogénéité des équations de champ; on voit mal ce qu'on a gagné à passer au cadre finslerien. L'influence du champ électromagnétique sur les trajectoires des particules matérielles électrisées et la déduction de ces trajectoires à partir des équations de champ ne sont pas mises en évidence. Le développement géométrique est par contre élégant. *A. Lichnerowicz* (Paris).

Narlikar, V. V., and Singh, K. P. Stationary gravitational fields. Bull. Calcutta Math. Soc. **43**, 168–174 (1951).

The authors examine several gravitational fields to see if they are stationary with respect to variations of the metric tensor which leave the stress-energy tensor unchanged. They show, in particular, that the external Schwarzschild solution is in this sense stationary for static, spherically symmetric variations. *A. Schild* (Pittsburgh, Pa.).

Raychaudhuri, Amalkumar. Arbitrary concentrations of matter and the Schwarzschild singularity. Physical Rev. (2) **89**, 417–421 (1953).

A cluster of particles moves radially under its own gravitational field with empty space outside. Using the line element

$$-e^\mu(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)+e^\nu dt^2,$$

with μ and ν functions of r and t, the author takes in the interior ($r<r_1$) $e^\nu=1$ and

$$e^\mu=e^q(1+\varepsilon r^2/4R^2)^{-2},$$

where $q=q(t)$, $\varepsilon=1$, 0, or -1, $R=$const. $>\frac{1}{2}r_1$. The corresponding energy tensor, calculated from the field equations, gives co-moving particles ($dr/dt=0$ for each) with a density ρ satisfying $\ddot{q}+\frac{1}{2}\dot{q}^2=-8\pi\rho/3$. Thus $\ddot{q}<0$, q cannot have a minimum, and so the system, once contracting, would go on contracting to infinite concentration. This interior solution is linked with continuity conditions across $r=r_1$ to an exterior solution for empty space ($r>r_1$). This exterior solution can be transformed into the Schwarzschild form with mass

$$m=(4\pi/3)\rho_1 e^{3q/2}(1+\varepsilon r_1^2/4R^2)^{-3}r_1^3,$$

ρ_1 being density at $r=r_1$. There are no singularities in the interior unless the concentration is infinite. There are formal singularities in the exterior, but one lies at infinite proper distance from the origin, and the other (making $g_{44}=0$) disappears on transformation to static form. The author dismisses these singularities as without physical reality, and concludes that he has a solution for the complete field of the cluster with no real singularity, whatever the concentration of the cluster may be. The last part of the paper deals with a spherically symmetric distribution of fluid matter. With the line element

$$-e^\lambda dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2)+e^\nu dt^2,$$

the field equations give $e^{-\lambda}=1-2m/r$ where

$$m(r)=4\pi\int_0^r T_4^4 r^2 dr,$$

and it is shown that $m(r_1)$ (the gravitational mass of the fluid bounded by $r=r_1$) remains constant in time if the pressure vanishes for $r=r_1$. *J. L. Synge* (Dublin).

Vaidya, P. C. The boundary conditions in gravitational fields of spherical symmetry. J. Univ. Bombay. Sect. A. (N.S.) **21**, no. 32, 1–7 (1952).

The author discusses non-static, spherically symmetric gravitational fields and, in particular, the conditions of fit at a spherical surface of discontinuity. *A. Schild.*

Vaidya, P. C. Radiation absorbing systems. Proc. Nat. Acad. Sci. India. Sect. A. **21**, 193–204 (1951).

The author studies a non-static spherically symmetric solution of the field equations of general relativity. The solution represents a radiation absorbing sink in a non-static homogeneous cosmological model. *A. Schild.*

Gürsey, Feza. Gravitation and cosmic expansion in conformal space-time. Proc. Cambridge Philos. Soc. **49**, 285–291 (1953).

A simple theory of gravitation is formulated in a conformally-flat Riemannian space-time. The perihelion precession for planetary motion is one half that predicted by general relativity and there is no gravitational bending of light rays. The theory is applied to an expanding steady-state universe with continuous creation of matter.

A. Schild (Pittsburgh, Pa.).

Jordan, P. Zur Integration der kosmologischen Gleichungen. Z. Physik **132**, 655–658 (1952).

The equations of the author's cosmological theory [Schwerkraft und Weltall, Grundlagen der theoretischen Kosmologie, Vieweg, Braunschweig, 1952] are ihtegrated approximately for the case of a quasi-linear expansion of the universe. This approximation is motivated by the recent revision of Hubble's constant which gives good agreement with an almost linear expansion. *A. Schild.*

Bose, S. N. Les identités de divergence dans la nouvelle théorie unitaire. C. R. Acad. Sci. Paris **236**, 1333–1335 (1953).

The author shows how to obtain, by use of a variational principle, the divergence-identities

$$g^{\mu\nu}g^{1/2}\frac{\partial R_{\mu\nu}}{\partial x^{\sigma}}-\frac{\partial}{\partial x^{\nu}}[(g^{\sigma\nu}R_{\mu\sigma}+g^{\sigma\nu}R_{\sigma\mu})g^{1/2}]=0$$

of the Einstein unified field-theory. No use is made of the equations relating the affine connection $\Gamma^{\alpha}_{\mu\nu}$ to the $g^{\mu\nu}$, which shows that the divergence-identities are independent of those equations. [Note: The reference to the Canadian J. Math. in the footnote on p. 1335 of the paper under review should be to vol. **2**, 1950, p. 125, equ. (17*b*), not to vol. **92**, 1924, p. 125, equ. (17*b*). The equation (17*b*) referred to is Einstein's own form of the above divergence-equation—see Canadian J. Math. **2**, 120–128 (1950); these Rev. **11**, 548.]

H. S. Ruse (Leeds).

Bergmann, Peter G., and Thomson, Robb. Spin and angular momentum in general relativity. Physical Rev. (2) **89**, 400–407 (1953).

Le problème d'une définition générale du moment angulaire d'un champ a été étudié par Rosenfeld, Belinfante, et Pauli dans le cadre de théories relativistes sans gravitation. Les auteurs se préoccupent ici d'atteindre moment angulaire et spin dans le cadre de la relativité générale classique ou, plus généralement, d'une théorie où le champ gravitationnel est représenté par un tenseur métrique symétrique $g_{\mu\nu}$. Sur la variété espace-temps, les auteurs introduisent l'espace fibré des repères orthonormés par rapport à cette métrique et une section locale de cet espace qui permet de définir un pseudo groupe de transformations de Lorentz locales. Ils étudient, par une technique de superpotentiels, une expression quasisymétrique pour l'impulsion-énergie, expression composée de deux parties: l'une ne faisant intervenir que des termes gravitationnels et qui n'est pas symétrique, l'autre, le "tenseur matériel", qui est covariante, symétrique et qui présente l'invariance de jauge et celle de spin. Cette expression est utilisée pour une discussion semiclassique du champ de l'électron de Dirac et elle conduit à une formule intéressante pour les composantes, par rapport aux repères introduits, du moment angulaire total du champ; le moment angulaire intrinsique de l'électron a bien la valeur attendue $\frac{1}{2}h$; mais le moment orbital diffère de l'expression classique par des termes gravitationnels qui traduiraient ainsi la courbure de l'espace au voisinage de l'électron.

A. Lichnerowicz (Paris).

Baccarani, Valeria, e Rubbiani, Franca. Sul problema della brachistocrona per un punto materiale veloce. Atti Sem. Mat. Fis. Univ. Modena **5**, 105–110 (1951).

Treats the problem of the brachistochrone for a particle with relativistic mass in a field of force depending only on position. The particular case of a constant force is worked out, the extremal curves being given in terms of elliptic integrals. *A. J. McConnell* (Dublin).

Pignedoli, Antonio. Sui moti tautocroni del punto materiale veloce. Atti Sem. Mat. Fis. Univ. Modena **5**, 72–82 (1951).

Discusses the one-dimensional tautochronous motion of a particle with relativistic mass, and the most general formula for the force, depending on position and velocity, is obtained for such a motion. A particular case is the force of elastic type, $F=-k^2x(1-v^2/c^2)^{-3/2}$, v being the velocity and x the distance from the centre. *A. J. McConnell* (Dublin).

Schlomka, Teodor. Zur Darstellung physikalischer und geometrischer Grössen durch Welttensoren 1. und 2. Stufe. Z. Naturforschung **7a**, 637–645 (1952).

The author studies tensor quantities in Minkowski space which, in a rest frame, represent preassigned tensors of 3-dimensional space such as mass, acceleration, current density, etc. *A. Schild* (Pittsburgh, Pa.).

Taylor, N. W. The relativistic electromagnetic equations in a material medium. Australian J. Physics **6**, 1–9 (1953).

This paper extends to material media in general relativity the method already described by the author for an electromagnetic field in vacuo [Australian J. Sci. Research. Ser. A. **5**, 423–429 (1952); these Rev. **14**, 806]. Tensor densities being indicated by Roman letters and tensors by italic, the basic equations are (*) $\partial\mathfrak{D}^{\mu\nu}/\partial x^{\nu}=\mathrm{J}^{\mu}$ where J^{μ} is the current vector density $-c^{-1}(j_1, j_2, j_3, ic\rho)$ and $\mathfrak{D}^{\mu\nu}$ is the tensor density derived from the tensor $D^{\mu\nu}$ whose components in the geodesic cartesian system (x, y, z, ict) are

$$D^{\mu\nu}=\begin{matrix} 0 & F_3 & -F_2 & N_1 \\ -F_3 & 0 & F_1 & N_2 \\ F_2 & -F_1 & 0 & N_3 \\ -N_1 & -N_2 & -N_3 & 0 \end{matrix}$$

where $F_1=iE_1-H_1$, $N_1=iD_1-B_1$, etc. Substitution in (*) gives the usual Maxwellian equations connecting **E**, **D**, **B**, **H**, **j**, ρ for the geodesic cartesian system. A skew-symmetric tensor $E^{\mu\nu}$ is introduced, its components in the geodesic cartesian system consisting entirely of the components of $i\mathbf{E}-\mathbf{B}$, and it is shown that there exists a tensor $B^{\mu\nu}$ such that

$$\mathrm{D}^{\mu\nu}=\mathrm{E}^{\mu\nu}+\mathrm{I}B^{\mu\nu}-\tfrac{1}{2}\epsilon^{\mu\nu\sigma\tau}\mathrm{R}B_{\sigma\tau},$$

where **I** and **R** mean that the imaginary and real parts are to be taken, retaining i in the imaginary parts. $B^{\mu\nu}=0$ in vacuo. The classical equations $\mathbf{D}=\epsilon\mathbf{E}$, $\mathbf{B}=\mu\mathbf{H}$ are replaced by

$$\mathrm{D}^{\mu\nu}+\tfrac{1}{2}\epsilon^{\mu\nu\sigma\tau}D_{\sigma\tau}=a\mathrm{R}E^{\mu\nu}+b\mathrm{I}E^{\mu\nu},$$

where a, b are two constants; this yields linear expressions for **H** and **D** in terms of **B** and **E**, resp. The ponderomotive force and the wave equation for vector and scalar potentials are derived. *J. L. Synge* (Dublin).

Gething, P. J. D. Rotation and magnetism in the world-models of kinematic relativity. Monthly Not. Roy. Astr. Soc. 112, 578–582 (1952).

From the author's summary: "Milne's theoretical relation between rotation and magnetism in the idealised world-models of Kinematic Relativity is examined. (The author's) results do not appear to support Milne's claim to have established a theoretical basis for a simple formula of the Blackett type."

A. Schild (Pittsburgh, Pa.).

MECHANICS

Novodvorskiĭ, E. P. On a method of design of mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **10**, no. 42, 5–51 (2 plates) (1951). (Russian)

One of the principal problems in the design of mechanisms is the approximation of desired functions. The techniques developed by Chebyshef and his followers are here summarized. A number of theorems are derived showing how the parameters of a considered mechanism are adjusted to obtain the greatest number of intersections of the traced function with the desired function and the least deviation between them. For several simple mechanisms, like the crosshead mechanism, sine and tangent mechanisms and the four-bar linkage, the exact formulas for the motions are given in terms of the parameters. Several numerical examples are given in detail showing how the approximation of certain desired functions by them is performed.

M. Goldberg (Washington, D. C.).

Pismanik, K. M. Design and analysis of hyperboloidal toothed gears. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **10**, no. 38, 27–58 (1950). (Russian)

After some remarks about the superiority of hyperboloid gears over the hypoid ones, the following is shown. Let I and II be two gears with skew axes, and let S be any moving surface (the tooth-cutting tool). The contact line of the envelope of S relative to I (and II) will be a straight line C (i.e., the teeth will have a full straight-line contact) if all the three relative screw velocities coincide at every moment, their axes coinciding with C. This implies that the absolute screw axis of S is fixed, and there are only three axodes. The axode I/II is a one-sheet hyperboloid of revolution, while the axode $S/I = S/II$ is a helicoid. Any line of the cylindroid $z = kxy/(x^2+y^2)$, where the z axis is the common normal of the axes I and II, is a possible absolute screw axis for S. The relations between the three absolute screw velocities and the dimensions of the axodes are derived, and the five motions and five settings necessary for the tooth-cutting lathe stated. The equations of the contact line of the teeth and of the tooth profiles are derived in parametric form.

A. W. Wundheiler (Chicago, Ill.).

Oliveri, E. Sul moto piano di un punto con accelerazioni radiale e trasversa proporzionali. Matematiche, Catania **7**, 55–61 (1952).

A particle moves in a plane, referred to polar coordinates, in such a way that the radial and transverse components of the acceleration are proportional. The author determines: (1) the motion, assuming that the trajectory is known; and (2) the trajectory, assuming that the angular coordinate is known as a function of time. Some simple examples are discussed in detail. *L. A. MacColl* (New York, N. Y.).

Gilvarry, J. J. Linear approximations in a class of non-linear vector differential equations. Quart. Appl. Math. **11**, 145–156 (1953).

This paper deals with a particle moving in three-dimensional space under a force which is the sum of two terms, the one the gradient of a scalar point function, the other an explicit function of time. The object of the author is to obtain linear differential equations which will determine a particular motion approximately in the neighborhood of the initial instant more satisfactorily than do the equations obtained by the usual process of linearizing the differential equations of motion.

Taking the tangent, principal normal, and binormal of the trajectory at the initial point as axes, he represents the coordinates of the typical point on the trajectory by the usual power series in the arc length s. Terms in s^n, $n>3$, are dropped, and the resulting equations are treated as a linear transformation from the variables s, s^2, s^3 to the coordinates x, y, z. The non-linear terms in the differential equations are expanded in power series in x, y, z, and then converted into power series in s by means of the transformation. Terms in s^n, $n>3$, are again dropped, and the expressions converted into linear expressions in x, y, z by means of the inverse of the transformation. In this way a system of linear differential equations is obtained, which the author regards as meeting his requirements.

It is difficult to judge the merits of this method. However, it appears to the reviewer that it is unlikely that the method will be really useful in any wide range of problems.

L. A. MacColl (New York, N. Y.).

***Loiseau, J. La mécanique rationnelle dans un espace à quatre dimensions et ses applications.** Publ. Sci. Tech. Ministère de l'Air, no. 270, Paris, 1952. viii+312 pp. 2,500 francs.

The author develops a complicated physical theory within the framework of four-dimensional Riemannian geometry. The theory is non-relativistic and preserves the Newtonian concept of an absolute time; space is curved and deforms with time. The chapter headings are: 1) General equations; 2) Electromagnetism—the electron; 3) Apparent variation of mass with velocity; 4) Gravitation; 5) Motion of a rigid body; 6) Study of a continuous medium; 7) Wave propagation; 8) Quanta and four-dimensional rational mechanics; 9) Quantum mechanics. There are appendices on exterior products and derivatives, and on some topics of differential geometry.

A. Schild (Pittsburgh, Pa.).

Cenov, I. On a new form of the equations of analytic dynamics. Doklady Akad. Nauk SSSR (N.S.) **89**, 21–24 (1953). (Russian)

Cenov, I. On some transformations of the equations of motion and on geodesic trajectories of mechanical systems. Doklady Akad. Nauk SSSR (N.S.) **89**, 225–228 (1953). (Russian)

Cenov, I. On Gauss's principle of least constraint. Doklady Akad. Nauk SSSR (N.S.) **89**, 415–418 (1953). (Russian)

If the partial derivatives are taken with t, q, $\dot{q}$, $\ddot{q}$, $\dddot{q}$ as independent variables, the equations of motion can be given the form (1) $R_i = \frac{1}{2}(\partial \ddot{T}/\partial \ddot{q}_i - 3\partial T/\partial q_i) = Q_i$. If there are nonholonomic constraints, $\dot{q}_i = a_{i\alpha}\omega_\alpha + a_i$, where ω_α are linear

in $\dot{q}_i$, the equations of motion are $a_{is}(R_i-Q_i)=0$. The author also derives a form of the equations with t, q, ω_s, $\dot{\omega}_s$, $\ddot{\omega}_s$ as independent variables. If T_0 is a value of T for some fixed values of $\dot{q}_i$, then $R_i=\partial R/\partial\ddot{q}_i$, $R=\frac{1}{2}(\ddot{T}-3\dot{T}_0)$. The merits of the form (1) are not discussed although their close relation to Appell's equations is mentioned. The author states (without detailed calculations) that $R-Q_i\ddot{q}_i$ is the Gaussian "constraint" $(\sum m(\ddot{x}-\ddot{x}_0)^2$, where $\ddot{x}_0$ is the vector acceleration without the constraints, and $\ddot{x}$ the one with them). It is difficult to see how this can be true with R depending on the superaccelerations $\dddot{q}$.

A. W. Wundheiler (Chicago, Ill.).

Ura, Taro. Sur les transformations canoniques exceptionnelles. Bull. Astr. **16**, 333–350 (1952).

An "exceptional" canonical transformation is one that preserves the Hamiltonian form of a particular Hamiltonian system even though it does not have this property for any Hamiltonian system. The author shows how the knowledge of a linear exceptional canonical transformation leads to a knowledge of two independent linear first integrals and conversely. The two body problem is given as an example.

D. C. Lewis, Jr. (Baltimore, Md.).

Mendes, Marcel. Transformations canoniques générales. C. R. Acad. Sci. Paris **236**, 457–458 (1953).

It is shown that a necessary and sufficient condition that a transformation $(q, p, t)\rightarrow(Q, P, T)$ applied to the system $\dot{q}_i=\partial H/\partial p_i$, $\dot{p}_i=-\partial H/\partial q_i$ should preserve the Hamiltonian form is that the differential $\sum_i(\lambda P_i dQ_i-p_i dq_i)+Hdt$ should be exact for $T=$constant. Here λ is a constant while P_i, p_i, and t are expressed in terms of q_i, Q_i, and T. The appropriate generalization to the case where the one independent variable t is replaced by several such independent variables is also made.

D. C. Lewis, Jr. (Baltimore, Md.).

Backes, F. Sur la méthode de la variation des constantes arbitraires dans les systèmes canoniques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 1051–1054 (1952).

The method of variation of arbitrary constants is established for canonical systems without making use of Poisson's bracket-expressions or contact transformations. It is based only on the knowledge of a complete integral of the Jacobi-Hamilton partial differential equation.

E. Leimanis.

Capon, R. S. Hamilton's principle in relation to non-holonomic mechanical systems. Quart. J. Mech. Appl. Math. **5**, 472–480 (1952).

The author shows explicitly by means of examples that, though some of the natural paths of a conservative non-holonomic system may satisfy Hamilton's principle, there may also be natural paths which do not. Here Hamilton's principle is understood in its strict sense whereby an integral $\int L(q, \dot{q}, t)dt$ is minimized relative to varied paths in which both the minimizing and varied paths satisfy the non-holonomic constraints. There is, of course, another principle leading to different equations (namely, the actual equations of motion) in which $\int Ldt$ is minimized relative to varied paths obtained from the minimizing path by displacements satisfying the constraints, but these varied paths do not themselves satisfy the constraints.

D. C. Lewis, Jr.

Nadile, Antonio. Equazioni miste del moto dei sistemi anolonomi. Boll. Un. Mat. Ital. (3) **7**, 302–306 (1952).

The author gives a form for the equations of motion of a conservative non-holonomic system which is particularly convenient in case there are so-called ignorable coordinates. The equations are the analogues of well-known equations given by Routh for holonomic systems.

D. C. Lewis, Jr.

Nadile, Antonio. Problemi dinamici dei sistemi anolonomi pei quali esiste un potenziale cinetico. Atti Sem. Mat. Fis. Univ. Modena **5**, 66–71 (1951).

The author studies some of the formalities of non-holonomic systems in which the forces are derived from a Lagrangian potential.

D. C. Lewis, Jr.

Nadile, Antonio. Forma sintetica delle equazioni del moto di un sistema anolonomo. Atti Sem. Mat. Fis. Univ. Modena **5**, 126–139 (1951).

A synthetic description of the equations of motion of a conservative non-holonomic system is given with the aid of the Riemannian geometry connected with the quadratic form representing the kinetic energy.

D. C. Lewis, Jr.

Zeuli, Tino. Sistemi dinamici corrispondenti con forze funzioni lineari delle velocità che ammettono la funzione lagrangiana. Atti Sem. Mat. Fis. Univ. Modena **5**, 146–153 (1951).

The following theorem is proved: Given two dynamical systems A and A_1 having forces linear in the velocities such that the trajectories of A and A_1 are identical although not necessarily the time laws of description. Then, if the system A is Lagrangian and if we limit attention to the trajectories of zero energy, the system A_1 is also Lagrangian, and the ratio of the two Lagrangian functions of these two systems gives the rate of change of the two time variables with respect to each other.

D. C. Lewis, Jr.

Grioli, Giuseppe. Questioni di dinamica del solido pesante asimmetrico. Rend. Sem. Mat. Univ. Padova **21**, 256–277 (1952).

The author gives a complete analytic description of the possible motions under gravity of an asymmetric rigid body fixed without friction at a point O, such that the moment of momentum with respect to O is the sum of two vectors, one of which has a fixed direction in space and the other has a fixed direction in the body. It is dynamically impossible for the magnitude of these vectors to be fixed, as the author proves.

D. C. Lewis, Jr. (Baltimore, Md.).

Sapa, V. A. Some cases of motion of a cylinder rotating about its axis. Izvestiya Akad. Nauk Kazah. SSR **1951**, no. 62, Ser. Mat. Meh. **5**, 154–167 (1951). (Russian)

Consider the motion of a cylinder, revolving about its axis and subjected to a translation whose initial velocity is perpendicular to the direction of the axis of the cylinder. Such a cylinder is then acted upon, in addition to the forces of gravity and resistance (assuming that the Newtonian square law holds, i.e., the resistance is proportional to the square of the linear velocity), by the Magnus force. The author considers only the cases when the equations of motion of the center of mass of the cylinder can be integrated in finite terms. In particular, he studies two cases: the axis of the rotating cylinder is (i) horizontal and (ii) vertical. In case (i) the following subcases are considered: (a) the resistance force is small in comparison with the force of gravity and the Magnus force, and (b) the resistance force is small in comparison with the Magnus force, and the modulus of the difference between the Magnus force and the resistance is small in comparison with gravity. In case (ii) the subcases considered are: (a) as in case (i), and (b)

the resistance, small in comparison with the Magnus force, is comparable to the force of gravity. The graphs of the trajectories described by the center of mass of the cylinder are sketched for various initial conditions. *E. Leimanis.*

Poritsky, H. Topics in gyroscopic motion. J. Appl. Mech. **20**, 1–8 (1953).

In an introduction the author reviews briefly the kinematical and dynamical concepts for the study of motion of a gyroscope with perfect alignment and balance. In addition to the fixed (x, y, z)-axes he introduces coordinate systems fixed to the rotor and to the inner and outer gimbals. The topics considered are the following: a) the erection of an electrically driven gyroscope when an electrical torque is applied; b) the effect of inequality of the principal moments of inertia normal to the spin axes; c) the effect of inertia of the gimbals. *O. Bottema* (Delft).

Hydrodynamics, Aerodynamics, Acoustics

Heinrich, G. Ergänzungen zu dem Aufsatz "Der Energietransport in strömenden Medien". Z. Angew. Math. Mech. **33**, 109–110 (1953).

The author answers objections made against an earlier paper [same Z. **32**, 286–288 (1952); these Rev. **14**, 509] by constructing a "rigorous" foundation for his energy transport vector. [His equations (5) and (6), on which his remarks are based, were given in slightly more general form by the reviewer, Physical Rev. (2) **73**, 513–515 (1948); these Rev. **9**, 474.] *C. Truesdell* (Bloomington, Ind.).

***Munk, W. H., and Arthur, R. S. Wave intensity along a refracted ray.** Gravity Waves, pp. 95–108. National Bureau of Standards Circular 521, U. S. Government Printing Office, Washington, D. C., 1952. $1.75.

This paper is concerned with the application of ray theory to the calculation of the intensity of waves refracted in an arbitrary, two-dimensional velocity field. A ray equation determines the path of the rays (orthogonals), and an equation of ray separation expresses the convergence or divergence along a ray. These two equations are combined into an equation of wave intensity from which the intensity along a ray can be determined given only the ray path and the wave velocity both on and near the ray. Application is made to special types of velocity fields. (From the authors' summary.) *J. V. Wehausen* (Providence, R. I.).

***Eckart, Carl. The propagation of gravity waves from deep to shallow water.** Gravity Waves, pp. 165–173. National Bureau of Standards Circular 521, U. S. Government Printing Office, Washington, D. C., 1952. $1.75.

Take the positive z-axis upwards and the undisturbed free surface at $z=0$, and let the pressure be written $-\rho gz+p(x, y, z, t)$. Let $p_0(x, y, t)=p(x, y, 0, t)$, $\kappa=\omega^2/g$ where ω is the frequency of a harmonic wave, $z=-h(x, y)$ be the equation of the bottom, and $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$. Assuming the usual linearized boundary condition on the free surface, the author derives the following integral equation:

$$p(z)=p_0e^{\kappa z}+\kappa^{-1}[1-\cosh \kappa z][\Delta p_0+\kappa^2 p_0]$$
$$+\kappa^{-2}\int_0^z [1-\cosh \kappa(\zeta-z)][\Delta^2+\kappa^2]\frac{\partial p(\zeta)}{\partial \zeta}d\zeta.$$

The term involving the integral is then neglected, but, as the author points out, without estimates to justify this. The resulting equation for p is then substituted into the boundary condition on the bottom, $p_xh_x+p_yh_y+p_z=0$, to give an equation for p_0 (where $\mu=1-e^{-2\kappa h}$, $k^2=\kappa^3 \coth \kappa h$):

$$\frac{1}{\mu}\left\{\frac{\partial}{\partial x}\left(\mu\frac{\partial p_0}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial p_0}{\partial y}\right)\right\}+k^2p_0=0.$$

$z=p_0/\rho g$ is an approximate equation for the free surface. The equation is analogous to that for shallow water waves. The solution of the approximate equation is compared in several cases with known exact solutions of the linearized equations. The agreement seems very close.

J. V. Wehausen (Providence, R. I.).

***Pierson, Willard J., Jr. On the propagation of waves from a model fetch at sea.** Gravity Waves, pp. 175–186. National Bureau of Standards Circular 521, U. S. Government Printing Office, Washington, D. C., 1952. $1.75.

Let $z=\eta(x, t)$ be the equation of the free surface of a two-dimensional gravity-wave in deep water. The author finds an explicit expression for $\eta(x, t)$ for three different choices of $\eta(0, t)$:

$$\eta_I(0, t)=A \exp\{-\sigma^2t^2\} \sin 2\pi t/T,$$

$$\eta_{II}(0, t)=\begin{cases} A \sin 2\pi t/T, & |t| \leq nT \\ 0, & |t|>nT, \end{cases}$$

$$\eta_{III}(0, t)=\sum_{n=-p}^{p} A_n \exp\{-\sigma^2(t-n\tau+\delta_n)^2\}$$
$$\times \sin\left\{\frac{2\pi}{T}(t-n\tau+\delta_n)+\theta_n\right\}.$$

Properties of the corresponding $\eta(x, t)$ are discussed in some detail and interpreted as possible mathematical models of the propagation of waves from a disturbed area in the ocean.

J. V. Wehausen (Providence, R. I.).

Birkhoff, Garrett. Formation of vortex streets. J. Appl. Phys. **24**, 98–103 (1953).

The vortex street behind a cylinder is discussed in detail in the light of the extensive theoretical and experimental literature of the subject and the following new theoretical points. (i) In two-dimensional incompressible flow, a distribution of vorticity along a straight segment cannot roll up into a circle of diameter less than 0.45 times the length of the original segment, unless additional energy is supplied to it from some outside source. (ii) Similarly, if two parallel vortex sheets roll up into a vortex street with the Karman spacing, the diameter of each vortex must be at least 0.26 times the distance between successive vortices in each row. (iii) An analysis of Strouhal number may be made in terms of the swinging of a portion of wake about the rear of the cylinder. Taking the force on it at $\pi\rho v^2\alpha$ per unit area (where α is its "angle of attack") ahd the volume as ρkd per unit area (where k is the "ratio of wake diameter to the cylinder diameter d") and the displacement as $l\alpha$, he deduces a simple harmonic oscillation with Strouhal number $\frac{1}{2}(d/\pi kl)^{\frac{1}{2}}$, and suggests values $k=1.33$, $l=1.5d$ to explain the typical Strouhal number 0.2 for intermediate Reynolds numbers. It is pointed out that l might reasonably be taken larger for the low Reynolds numbers, so that the lower frequency then observed is ascribed to greater wake inertia. (iv) The moment of the vorticity in a plane viscous flow remains constant, and for a periodic flow this is true for each period.

Hence as the total vorticity per period decreases by diffusion, the distance between the vortex rows must increase. (v) From arguments for an inviscid fluid, the spacing-ratio will tend to remain constant, except for the slow increase resulting from (iv). *M. J. Lighthill* (Manchester).

Dorfman, L. A. Computation of irrotational flow about grids of profiles and construction of grids for given velocity distribution on the profiles. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 599–612 (1952). (Russian)

Let $z(\zeta)$ map the grid of flat plates at $\eta=m\pi i$ $(m=0, \pm1, \pm2, \cdots)$, $|\xi|\leq a$, in the $\zeta=\xi+i\eta$ plane onto a given grid of congruent equally spaced airfoils in the $z=x+iy$ plane. The potential function for incompressible flow about plane grids being known, the potential for the given grid can be found by constructing $z(\zeta)$ as follows. For a periodic analytic function $F(\zeta)$ bounded at infinity the limits $F_+(\xi_0)$ and $F_-(\xi_0)$ obtained by approaching the point $\zeta=\xi_0$ of s slit satisfy

$$\pi i[F_+(\xi_0)+F_-(\xi_0)]=\int_{-a}^{a}[F_+(\xi)-F_-(\xi)]\coth(\xi-\xi_0)d\xi,$$

which can also be inverted to yield F_+-F_- in terms of F_++F_-. Application of these relations to $z(\zeta)-\zeta$ yields

$$2\pi[x_\pm(\xi_0)-\xi_0]=\int_{-a}^{a}[y_+(\xi)-y_-(\xi)]\coth(\xi-\xi_0)d\xi$$
$$\pm(\cosh^2 a-\cosh^2\xi_0)^{\frac{1}{2}}\int_{-a}^{a}[y_+(\xi)+y_-(\xi)]$$
$$\times(\cosh^2 a-\cosh^2\xi)^{-\frac{1}{2}}/\sinh(\xi-\xi_0)d\xi$$

and $y_\pm$ can be expressed similarly in terms of $x_\pm$. The author suggests solving these equations by an iterative scheme. The integrations are performed numerically after making the transformation $\tanh\xi=\tanh a\cos\theta$ and approximating integrands by trigonometric polynomials. A scheme devised by L. Simonov [same journal **11**, 69–84 (1947); these Rev. **9**, 541] for airfoil computations is applied to accelerate convergence. Calculations for a particular profile show good agreement with test data. The author also discusses specializations for grids of high solidity and the construction of grids with velocity distributions prescribed as functions of arc length on the profiles. *J. H. Giese.*

Couchet, G. Sur la force vive d'un fluide et le calcul des efforts de ce fluide sur un profil en mouvement (fluide parfait, incompressible, en repos à l'infini). Recherche Aéronautique no. **32**, 11–13 (1953).

Tsuji, Hiroshi. Note on the solution of the unsteady laminar boundary-layer equations. J. Aeronaut. Sci. **20**, 295–296 (1953).

In this note, the author points out that, in the problem considered by Moore [NACA Tech. Note no. **2471** (1951); these Rev. **13**, 401], if $(X/U)(\partial\zeta_0/\partial T)$, X, U, T being the distance along the body, forward speed, and time, respectively, is a polynomial in ζ_0, then

$$\zeta_n\ [=(X^{n+1}/U^{n+2})d^{n+1}U/dT^{n+1},\quad n=0, 1, 2, \cdots]$$

is also polynomial of ζ_0. Under this condition, the equations can be reduced to systems of total differential equations. *Y. H. Kuo* (Ithaca, N. Y.).

Murphy, James S. Some effects of surface curvature on laminar boundary-layer flow. J. Aeronaut. Sci. **20**, 338–344 (1953).

Under the assumption of moderate and large curvature of the solid surface, the modified boundary layer equations for both cases are derived together with the required boundary conditions. To study the effect of curvature on the skin friction, the author considers a very artificial surface whose curvature varies as $x^{-1/2}$, x being distance measured along the surface, and whose surface pressure is uniform. If this is granted, then the author shows that the skin friction is above or below the flat plate value for concave or convex curvature respectively. *Y. H. Kuo* (Ithaca, N. Y.).

Illingworth, C. R. The laminar boundary layer of a rotating body of revolution. Philos. Mag. (7) **44**, 389–403 (1953).

The author, by combining the Mangler and the von Mises transformations, reduces the problem of boundary-layer flow around a rotating body of revolution to the one of an incompressible plane flow. Solutions in series-form for a circular cone at supersonic forward speed have been obtained. The results show that the effect of spin on drag and torque in most of the ballistic problems is small. From this conclusion, the author further simplifies the problem for slow rotation under the assumptions: (1) Prandtl number is one; (2) viscosity is proportional to the absolute temperature; and (3) the wall is thermally insulated.
Y. H. Kuo (Ithaca, N. Y.).

Zondek, B., and Thomas, L. H. Stability of a limiting case of plane Couette flow. Physical Rev. (2) **90**, 738–743 (1953).

The authors consider a parallel flow with a linear velocity distribution in the upper half-plane. The stability of this flow is proved for all small wavy disturbances. This also tends to confirm the usual belief (which has, however, never been rigorously proved) that the plane Couette flow between two planes is stable. *C. C. Lin* (Cambridge, Mass.).

***v. Krzywoblocki, M. Z. On the generalized fundamental equations of isotropic turbulence in compressible fluids and in hypersonics.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 827–835. The American Society of Mechanical Engineers, New York, N. Y., 1952.

This paper presents a straightforward generalization to a perfect gas of the methods which have been developed by G. I. Taylor, von Kármán and Howarth, H. P. Robertson, and others for treating isotropic turbulence in an incompressible fluid. By introducing a large number of suitable correlations between the pressures, temperatures, densities, and velocity components at two different points in the gas, the author derives from the equations of continuity, motion and total energy, a number of equations for the defining scalars of the various isotropic tensors which are introduced. [Related earlier papers of the author are J. Franklin Inst. **254**, 317–322 (1952); J. Phys. Soc. Japan **7**, 299–300 (1952); these Rev. **14**, 596.] *S. Chandrasekhar.*

Obukhoff, A. M., and Yaglom, A. M. The microstructure of turbulent flow. NACA Tech. Memo no. **1350**, 41 pp. (1953).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 3–26 (1951); these Rev. **12**, 873.

Parker, Eugene N. Extension of Heisenberg's model of turbulence to critical Reynolds numbers. Physical Rev. (2) **90**, 221–223 (1953).

The author applies Heisenberg's concept of fully developed turbulence to the problem of initiation of turbulence, and obtains the critical Reynolds number of laminar instability after a number of approximations. The results are claimed to be in agreement with established values. However, the thickness of the boundary layer used in the present paper differs from the displacement thickness usually used by a factor of about three; thus the agreement is actually much worse.

C. C. Lin (Cambridge, Mass.).

***Frenkiel, F. N. Turbulent diffusion from a non-punctual source.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 837–841. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The author calculates the density distribution after some "dispersion time" t for an initial distribution which is uniform within a sphere of radius a. The assumptions used in the calculation are as follows: (1) The distribution due to a point source is Gaussian with a standard deviation depending on time and the Lagrangian correlation coefficient. (2) The distribution due to a distributed source may be obtained from that of a point source by superposition. No discussion is given to show whether these two assumptions are consistent. (It is known to be so when the coefficient of diffusion is a constant, but this is no longer obvious when variable diffusion coefficient is involved.) *C. C. Lin.*

Pearson, Carl E. Note on self-propagation of turbulent spots. Quart. Appl. Math. **11**, 219–222 (1953).

The author discusses the self-propagation of turbulent spots under two hypotheses. It is shown that the same asymptotic shape is obtained under either one.

C. C. Lin (Cambridge, Mass.).

Gilbarg, David. Comparison methods in the theory of subsonic flows. J. Rational Mech. Anal. **2**, 233–251 (1953).

Dans des travaux récents l'auteur et J. B. Serrin ont utilisé avec succès le principe du maximum pour obtenir d'une façon fort simple des théorèmes de comparaison pour la vitesse sur la ligne libre des écoulements irrotationnels, plans ou de révolution, des fluides incompressibles. Dans le présent travail l'auteur étend cette méthode au cas des mouvements subsoniques et à potentiel des fluides compressibles; ces résultats reposent sur le fait que la différence de deux fonctions de courant, correspondant à des écoulements de ce type définis dans une même région, vérifie une équation du type elliptique qui satisfait au principe du maximum. Il est montré en particulier que dans le cas d'un écoulement le long d'un mur la vitesse en un point du mur est une fonction monotone de la vitesse à l'infini. Signalons aussi un théorème d'unicité pour les écoulements rencontrant un obstacle symétrique, fini. Ces résultats sont étendus aux écoulements dans un canal et pour terminer l'auteur envisage les mouvements méridiens, pour lesquels reste valable le théorème sur la dépendance monotone de la vitesse en un point de la frontière.

R. Gerber (Toulon).

Ludford, G. S. S. D'Alembert's paradox and the moment formula in three-dimensional subsonic flow. Rev. Fac. Sci. Univ. Istanbul (A) **18**, 1–13 (1953). (Turkish summary)

The author proves the D'Alembert's paradox by finding the first term of the asymptotic solution to the differential equation for the velocity potential of a pure subsonic flow around a body. The moment integral is also evaluated and the result agrees with that of the linearized theory.

Y. H. Kuo (Ithaca, N. Y.).

Ghaffari, A. Sur les solutions asymptotiques d'écoulement compressible subsonique. Bull. Sci. Math. (2) **76**, 112–118 (1952).

Survey article. *M. J. Lighthill* (Manchester).

***Garrick, I. E. On moving sources in nonsteady aerodynamics and in Kirchhoff's formula.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 733–739. The American Society of Mechanical Engineers, New York, N. Y., 1952.

Kirchhoff's formula, expressing the solution to the classical wave equation in terms of a surface distribution of sources and doublets, is formulated in a moving coordinate system and the application to nonsteady motion of thin wings in subsonic, transonic, and supersonic flow discussed.

J. W. Miles (Los Angeles, Calif.).

Germain, Paul, et Liger, Marc. Une nouvelle approximation pour l'étude des écoulements subsoniques et transsoniques. C. R. Acad. Sci. Paris **234**, 1846–1848 (1952).

Continuing the work of a previous note [Germain and Fenain, same C. R. **234**, 592–594 (1952); these Rev. **13**, 793] the authors reduce the study of the equation (*) $k(\sigma)\psi_{\sigma\sigma}+\psi_{ss}=0$ defining the flow of a fluid to the study of the Tricomi equation; $su_{zz}+u_{ss}=0$. This is accomplished by the transformation $3y=2z^{3/2}$, $x+iy=\tan(\theta/a+is)$, $\beta(s)\psi(\theta,s)=y^{1/6}u(z,x)$, where a is a constant. If $\beta(s)$ satisfies the equation $9(\sinh^2 2s)\beta''+5\beta=0$, then a 1-1 correspondence is set up between solutions of (*) and the Tricomi equation. With the aid of this correspondence some known facts for the Tricomi equation may be employed to obtain gas-dynamical results. As examples the authors discuss critical jet flows and the transonic flow about a profile.

M. H. Protter (Syracuse, N. Y.).

Sasaki, Tatudiro. On the solution of trans-sonic flow using Laplace transformation. J. Phys. Soc. Japan **7**, 625–626 (1952).

The essence of the author's result is that if $\psi(q,\theta)$ is the stream function of a two-dimensional irrotational incompressible flow, expressed in terms of the fluid speed q and the flow direction θ, then

$$\psi - \frac{q^2}{4a_0^2}\frac{\partial\psi}{\partial q}$$

is an approximation to order M^2 (i.e., neglecting terms in M^4, where M is the Mach number) for the stream function of a corresponding compressible flow in which a_0 is the velocity of sound at a stagnation point. The result is clearly connected with the fact that the Kármán-Tsien solution is valid to order M^2 for arbitrary adiabatic index, but the reviewer knows of no explicit previous statement of it. The result is applied to obtain an approximate transonic flow past a nearly circular cylinder. *M. J. Lighthill.*

Ribner, Herbert S., and Arnoff, E. Leonard. Interaction between a supersonic stream and a parallel subsonic stream bounded by fluid at rest. NACA Tech. Note no. 2860, 45 pp. (1952).

Tsien and Finston's model of a shock-wave boundary-layer interaction [J. Aeronaut. Sci. 16, 515–528 (1949); these Rev. 11, 554] is altered by applying a condition of uniform pressure, instead of a condition of flow along a solid wall, at the inner boundary of the uniform subsonic layer which is used to replace the boundary layer. The results show even less diffusion of the incident wave in the boundary layer than in the Tsien and Finston model. It is concluded that when a shock wave is incident on a laminar layer, so that the layer separates and in the steady state the shock is incident on a dead air region at nearly uniform pressure, the separated boundary layer will turn through the expected angle at the point of incidence in a distance negligible for practical purposes. *M. J. Lighthill.*

Cherry, T. M. A transformation of the hodograph equation and the determination of certain fluid motions. Philos. Trans. Roy. Soc. London. Ser. A. **245**, 583–626 (1953).

A transformation is given of the hodograph equations of two-dimensional gas dynamics from the usual variables q, θ to q and a new variable ϕ. The transformation, which suits any gas with pressure-density relation $p\rho^{-\gamma}=\text{const.}$, $\gamma>1$, is particularly useful for (i) transonic nozzle flow, and (ii) flow past a fixed cylinder in a uniform subsonic stream. In each of these cases a solution, which in terms of q, θ is necessarily multiple-valued, becomes a single-valued function of q, ϕ. The solution may thus be represented over the whole domain of interest by a single series in q, ϕ, whereas, ordinarily, different series would be required for different branches of the function [see, e.g., Goldstein, Lighthill, and Craggs, Quart. J. Mech. Appl. Math. 1, 344–357 (1948); these Rev. 10, 641].

The author constructs a certain linear combination of the set of "Chaplygin solutions", and claims for this solution a fundamental importance. Indeed, in the variables q, ϕ it has a wide domain of regularity; moreover, related solutions turn out to give a special flow of type (i), and a family of flows of kind (ii) for which the cylinders are aerofoil-shaped. Complete numerical results are given for the first case. These are used to check the accuracy of some approximate nozzle theories. In the second case it seems not improbable that some flows will exhibit a supersonic region without a corresponding limit line. *J. B. Serrin.*

Hasimoto, Zirō. A note on cross Mach lines in simple waves in the plane supersonic flow. J. Phys. Soc. Japan 8, 248–251 (1953).

The author finds general representations in simple form and applicable to an arbitrary barotropic gas for the cross Mach lines and the value of the stream function along any cross Mach line of a simple wave in a steady, supersonic, two-dimensional, irrotational, isentropic flow. *P. Chiarulli* (Providence, R. I.).

Hasimoto, Zirō. On the limiting line in a simple wave. J. Phys. Soc. Japan 8, 251–254 (1953).

The envelope of the straight Mach lines of a simple wave in a steady, supersonic, two-dimensional, irrotational, isentropic flow of a perfect gas is called a "limiting line" and it is shown that the cross Mach lines have a cusp and the distribution of the stream function along the cross Mach line takes an extremum at the envelope. The proof of the "cusping" is not complete in that though the first differentials along the cross Mach lines are shown to vanish at the envelope, no mention is made of the behavior of higher differentials. *P. Chiarulli* (Providence, R. I.).

Tamada, Kō. On the detachment of shock wave from the leading edge of a finite wedge. J. Phys. Soc. Japan 8, 242–247 (1953).

The author studies the flow behind the shock wave attached to the leading edge of a finite wedge placed in a supersonic stream at zero angle of attack for that range of free stream velocities for which this flow is subsonic. It is assumed that the flow and shock are only slightly perturbed from the uniform flow behind the attached uniform shock that results in the case of the infinite wedge and, furthermore, that the field of flow is still irrotational and isentropic. This linearized problem is solved approximately in that a particular form of velocity potential and shock position is assumed to hold. *P. Chiarulli* (Providence, R. I.).

Cabannes, Henri. Calcul de la courbure au sommet de l'onde attachée dans les écoulements plans non stationnaires. J. Rational Mech. Anal. 2, 219–232 (1953).

The unsteady flow of two-dimensional compressible fluid in the neighbourhood of the apex of an obstacle is examined by a perturbation method. Polar coordinates (r, θ) are taken with the apex as origin, and t is the time; the physical variables are expanded in a series of ascending powers of r with coefficients which are functions of θ and t. The first term in each case refers to a flow which, at any instant, is a uniform stream. Equations are derived which give the next approximation.

The method is applied to the flow near the apex of a two-dimensional body in a uniform supersonic stream, the body turning about its leading edge in a manner arbitrarily prescribed except that the shock-wave must remain attached. Throughout the paper the shock-wave is assumed to be analytic. An expression is found for the ratio of the radii of curvature of the shock-wave and the body at the apex. This ratio may take positive or negative values or both, depending upon the motion. The results reduce to those of L. Crocco [Aerotecnica 17, 519–534 (1937)] for a body at rest (steady flow). At "Crocco's angle" the curvature of the shock-wave at the apex is infinite for all but one angular velocity of rotation. It is also infinite at the limiting angle at which the shock-wave detaches itself from the wedge. For the limiting angle in steady flow Crocco found that the shock-wave had finite curvature of opposite sign to that of the body, a difference worthy of remark. *D. C. Pack.*

Thomas, T. Y. The separation of supersonic flow from curved profiles. Indiana Univ. Publ. Sci. Ser. no. 17, 69 pp. (1951).

The theory of simple waves and its use in the standard calculation of supersonic flow around an aerofoil is first explained in detail. This theory is inadequate after and immediately ahead of the point at which the flow separates from the profile. The author makes certain postulates about the occurrence of separation and the shock-wave at the rear of the profile which serve to fix the point of separation and the subsequent flow in the vicinity of the aerofoil. By terminating the expansion obtained from the standard calculation and continuing with a simple wave of compression there is found just one continuous distribution of pressure ahead of the separation point which gives, at the separation point, the conditions determined from the postu-

lates. Calculations have been carried out for two aerofoils for which experimental results are available; they lead to very satisfactory reproduction of the pressure distributions and, in particular, give closely the location of the region of back pressure and of the separation point. *D. C. Pack.*

Ribner, H. S. Convection of a pattern of vorticity through a shock wave. NACA Tech. Note no. **2864**, ii+48 pp. (1953).

As a step towards the problem of determining what happens when a strong shock wave passes through a turbulent flow, the problem is considered where the turbulent flow is replaced by a sinusoidal uni-directional shear flow arbitrarily inclined to the plane of the shock wave. This problem is then reduced by a change in frame of reference to a two-dimensional steady-flow problem and treated by an adaption of Sears's [J. Math. Physics **28**, 268–271 (1950); these Rev. **11**, 625] non-isentropic linear perturbation theory. The author finds a non-uniform pressure field behind the shock wave. In the original unsteady flow problem this corresponds to sound waves radiated as the shear pattern passes through the shock wave. Rough conclusions are drawn concerning the sound radiated when turbulence passes through a shock wave, and it is pointed out that this may be an important source of sound in wind-tunnels.

M. J. Lighthill (Manchester).

Pistolesi, E. Confronto fra alcuni metodi dell'aerodinamica supersonica. Aerotecnica **32**, 242–245 (1952).

Examples are given to show that the so-called "doublet" method of solving "inverse problems" in supersonic lifting surface theory ("given the lift distribution, find the downwash distribution") may lead to error unless handled with care. The point made seems to be that the doublet strength must be made to vanish on any "subsonic" edges of the wing. The problems treated are the delta wing in direct and reverse flow with uniform lift distribution. The method recommended in preference to the doublet method is described as the "method of vortex elements".

M. J. Lighthill (Manchester).

Thom, A., and Klanfer, Laura. The method of influence factors in arithmetical solutions of certain field problems. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2440** (**9854**, **11,010**), 30 pp. (1953).

"This paper gives an extension to the squares method of solving certain field problems. The idea of influence factors is used." In particular, the problem of computing two-dimensional compressible flows about a sharp-edged airfoil in a channel is treated in great detail. The method used is to expand in powers of the upstream Mach number about the incompressible flow solution. The value of the solution at "infinity" is approximated by using the concept of "influence factor," thus eliminating a considerable amount of the work of solving the finite difference equations.

"The use of the (influence) factor F developed in this paper is thus fully justified in that it shortens enormously the work involved in obtaining solutions to the problem of the flow in a channel attacked by the potential function method. Perhaps the most important result obtained by the application of the method is that the blockage effect is not uniformly distributed along the aerofoil, but causes a much greater increment at the maximum ordinate than at the ends. Another conclusion is that the mass flow method . . . gives reasonably good results."

A bibliography, many figures, and tables of illustrative computations are included. *E. Isaacson.*

Marini, M. Il calcolo delle forze aerodinamiche d'inerzia sulle ali a freccia. Aerotecnica **32**, 354–361 (1952).

Inertia derivatives are calculated for the rigid symmetrical motion of a rectangular wing and of a straight swept back wing of constant chord. The aspect ratio is G in both cases, and the angle of sweepback in the second case is 45°. The author follows W. P. Jones [Jones and Skan, Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **1900**, 29–66 (1949); Jones, ibid. no. **1947** (1941)] in replacing the wing by a doubled distribution, such that the chordwise distribution is taken from two-dimensional theory. A familiar expansion is then assumed for the spanwise distribution and its coefficients are determined by collocation. As usual, the formulae for the induced normal velocities lead to improper integrals. The singularities are isolated and the resultant expressions are calculated partly by analytical and partly by numerical methods.

A. Robinson (Toronto, Ont.).

***Fettis, H. E. An approximate method for the calculation of non-stationary air forces at subsonic speeds.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 723–732. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The Possio integral equation for the pressure distribution on a two-dimensional, oscillating airfoil in subsonic, compressible flow is attacked by approximating the non-singular portion of the kernel as a linear combination of the incompressible kernel and a polynomial, the coefficients of the latter being determined by a least squares comparison between the approximate and exact kernels. A solution to the approximate integral equation is then constructed along the lines of Schwartz' solution for the incompressible problem [Luftfahrtforschung **17**, 379–386 (1940); these Rev. **3**, 286]. *J. W. Miles* (Los Angeles, Calif.).

Ruppel, Werner, et Weber, Robert. Le calcul de flutter en régime supersonique. Z. Angew. Math. Physik **4**, 128–145 (1953).

***Serruys, Max. Etude générale de l'écoulement d'un gaz à travers une tuyère quelconque et du passage par la vitesse du son (en régime permanent ou quelconque avec apport de chaleur et réaction chimique éventuels).** Publ. Sci. Tech. Ministère de l'Air, no. **272**, Paris, 1952. viii+62 pp. 750 francs.

This report gives an up-to-date account of the problem of one-dimensional steady and unsteady flow of compressible fluids through a pipe under the most general conditions, such as with or without heat addition and with or without change of chemical compositions. It consists of four parts. Part 1 states briefly the purpose of the present study. Part 2 is devoted to the general subject: "the study of the flow phenomena by thermodynamics and classical fluid mechanics." The differential equations involved in the problem are derived and discussed. Part 3 specializes on the steady flow problems. Still in general terms, the integrals of the differential equations are available either in integrated form or in the form of quadratures. Part 4 entitles conclusions which recapitulate the important results. *Y. H. Kuo.*

Parker, Eugene M. Acoustical radiation from the velocity field in a compressible fluid. Physical Rev. (2) **90**, 240–242 (1953).

The author attempts to obtain a solution of the problem of the sound radiation field from a bounded region of fluctuating fluid flow. This is the problem studied by the reviewer [Proc. Roy. Soc. London. Ser. A. **211**, 564–587 (1952); these Rev. **13**, 879], but the author appears to claim that his approach is physically more lucid and mathematically less complicated. The reviewer may reasonably be accused of prejudice, but is obliged to state that he disagrees with both claims. He is unable to check whether the answers given by the two methods are the same (as the author claims) because of his imperfect comprehension, after several readings, of the meaning of the symbols introduced. *M. J. Lighthill* (Manchester).

Eckart, Carl. The theory of noise in continuous media. J. Acoust. Soc. Amer. **25**, 195–199 (1953).

For random sound fields ("noise" in perhaps its original sense!) the properties of the function

$$\psi(\mathbf{x}_1, \mathbf{x}_2, \tau) = \overline{p(\mathbf{x}_1, t)p(\mathbf{x}_2, t-\tau)}$$

(where $p(\mathbf{x}, t)$ is the pressure at $\mathbf{x}$ at time t) are discussed in some detail, including its special properties when "uniformity" (what in turbulence theory is called "homogeneity"), with or without isotropy, is assumed. The character of the anisotropy of noise near a reflecting plane is elucidated. The paper concludes with a discussion of the noise generated by random motions on an infinite plane bounding surface. *M. J. Lighthill* (Manchester).

Haranen, V. Ya. On the propagation of sound in a medium with random fluctuations of the refractive index. Doklady Akad. Nauk SSSR (N.S.) **88**, 253–256 (1953). (Russian)

L'auteur considère le problème de la propagation du son lorsque les variations du coefficient de réfraction n sont dues au hasard; n est une fonction continue stationnaire et les caractéristiques du milieu ne dépendent pas de la direction—une isotropie macroscopique. Si φ est l'angle aléatoire que fait le rayon avec un plan tangent P à l'origine, après un parcours s, l'auteur trouve que

$$\overline{\varphi^2} = 2s\frac{\overline{(\text{grad } n)^2}}{3n^2}\int_0^\infty R(p)dp$$

où $R(s_2-s_1)$ est le coefficient de l'autocorrélation de la fonction $d\varphi/ds$.

Si l'on suppose que les intervalles élémentaires Δs sont suffisamment grands pour que les $\Delta\varphi$ soient statistiquement indépendants, le processus de variation devient un processus du type des chaînes de Markoff, et la densité de probabilité w est donnée par l'équation: $\partial w/\partial s - D\partial^2 w/\partial\varphi^2 = 0$ où $D = \overline{\varphi^2}/2s$ dont la solution est une distribution de Gauss. En désignant par z la distance du rayon au plan P, après un parcours s, c'est-à-dire $z = \int_0^s \sin\varphi(s)ds$, on trouve: $\overline{z^2} = 8Ds^3$, une formule qui peut être très utile dans ses applications; par exemple, la propagation du son dans la couche de la mer au voisinage de sa surface libre. L'auteur étudie ensuite quelques applications de propagation dans une couche limitée par deux plans parallèles. *M. Kiveliovitch.*

Keller, Joseph B. Finite amplitude sound waves. J. Acoust. Soc. Amer. **25**, 212–216 (1953).

L'auteur forme les solutions exactes des équations du mouvement par tranches planes d'un gaz dont l'équation d'état est de la forme $p = a(s) - k^2(s)\tau$ où p, τ, s, désignent la pression, l'entropie, et le volume spécifique. On sait d'ailleurs que toute équation d'état peut être approchée localement par une équation de ce type. Les solutions indéfinies des équations envisagées correspondent à des mouvements par ondes continues, pourvu que les variations de pression ne soient pas trop grandes. Lorsque ces variations dépassent une certaine limite, il existe des chocs ou des cavitations. Une limite de l'amplitude des ondes pour éviter ces phénomènes est ainsi obtenue; celle-ci correspond, dans le cas d'un choc, à un déplacement des particules égal à $\lambda/2\pi$, où λ désigne la longueur d'onde.

Deux problèmes particuliers sont étudiés en détail: celui des ondes progressives engendrées par le mouvement vibratoire d'un piston et celui des ondes stationnaires dans un tube fermé. Pour ces deux cas sont données les courbes représentatives des vitesses et des volumes spécifiques en fonction du temps ou de la position. On voit en particulier que pour l'onde progressive la vitesse en un point est dirigée vers le piston pendant plus de la moitié de la période. *R. Gerber* (Toulon).

Karal, F. C. The analogous acoustical impedance for discontinuities and constrictions of circular cross section. J. Acoust. Soc. Amer. **25**, 327–334 (1953).

Es handelt sich um die Aufgabe der Bestimmung der akustischen Impedanz für abrupte Aenderungen des kreisförmigen Querschnittes einer Röhre. Nach einer Erörterung des Impedanzbegriffes für diesen Fall werden Ausdrücke für den Schalldruck und für die Schallgeschwindigkeit des Schallfeldes im genannten Falle abgeleitet. Diese Ausdrücke führen dann unmittelbar auf Formeln für die aequivalente Impedanz der Discontinuität. Diese Formeln werden anhand einer Kurve numerisch erörtert. *M. J. O. Strutt.*

Guptill, Ernest W. The sound field of a piston source. Canadian J. Physics **31**, 393–401 (1953).

Verf. behandelt das Schallfeld zwischen zwei parallelen, unendlich grossen, ebenen starren Wänden, wenn in einer dieser Wände eine runde Kolbenmembran befindet. Für das Geschwindigkeitspotential wird ein unendlicher Integralausdruck aufgestellt mit Koeffizienten, welche durch die Randbedingungen bestimmt werden. Nach einer Diskussion der Integrationswege gelangt Verf. zu einer Lösung, welche graphisch als Funktion des gegenseitigen Abstandes der Wände im Verhältnis zur Wellenlänge dargestellt wird. Es wird auch der Fall erörtert, dass bei der Kolbenmembran die Amplitude eine Funktion des Abstandes vom Zentrum ist. *M. J. O. Strutt* (Zürich).

Miles, John W. Transient loading of a baffled piston. J. Acoust. Soc. Amer. **25**, 200–203 (1953).

Verf. betrachtet eine Kreiskolbenmembran, welche in einer unendlich grossen, ebenen, starren Wand sich bewegt. Die erste Aufgabe besteht in der Bestimmung der erforderlichen angewandten Kraft, um eine Geschwindigkeit der Kolbenmembran zu erzeugen, welche zur Zeit $t=0$ plötzlich von Null auf eins wächst. Dieses Ergebnis erlaubt dann, durch Anwendung des Ueberlagerungssatzes von Duhamel, die Bestimmung der Kraft zur Erzielung irgend einer gegebenen Membrangeschwindigkeit als Funktion der Zeit. Die zweite Aufgabe besteht darin, dass die Membrange-

schwindigkeit bestimmt wird, welche eine Kraft erzeugt, die zur Zeit $t=0$ plötzlich von Null auf eins wächst. Zur Lösung dieser Aufgaben betrachtet Verf. die Randbedingungen und die Differentialgleichung der Aufgabe. Die Lösung der zweiten genannten Aufgabe wird graphisch dargestellt.

M. J. O. Strutt (Zürich).

Miles, John W. Transient loading of a baffled strip. J. Acoust. Soc. Amer. **25**, 204–205 (1953).

Verf. betrachtet einen unendlich langen, ebenen Kolbenstreifen gleichbleibender Breite, der von einer unendlich grossen, ebenen, starren Wand umgeben ist. Es handelt sich um die Bestimmung der auf den Streifen wirkenden Kraft, welche einen Einheitssprung der Geschwindigkeit als Funktion der Zeit erzeugt. Die Lösung wird aus dem Geschwindigkeitspotential des Schallfeldes durch eine Laplacesche Transformation ermittelt. Das Ergebnis wird graphisch dargestellt und numerisch diskutiert. *M. J. O. Strutt.*

Gerjuoy, Edward. Total reflection of waves from a point source. Comm. Pure Appl. Math. **6**, 73–91 (1953).

In this paper an analysis is made of the reflection of a pulse of sound from a point source in a non-absorbing liquid plane boundary, under the circumstance that the index of refraction is <1. The analysis is equally applicable to the total reflection of electromagnetic radiation from a vertical dipole, in the absence of absorption. The results clarify the mechanism of propagation of the head wave which has been observed in acoustic experiments. It is shown without approximation that a pulse is received in the first medium at a time corresponding to propagation along the surface with wave velocity of the second medium. *A. E. Heins.*

Eckart, Gottfried. Etude des échos des ondes acoustiques dans le milieu stratifié de la troposphère. Acustica **2**, 256–262 (1952).

Sound reflection by a stratified atmosphere is studied by using the techniques of Bremmer [Philips Research Rep. **4**, 1–19, 189–205 (1949); Physica **15**, 593–608 (1949); these Rev. **10**, 656; **11**, 142, 631] and Schelkunoff [Quart. Appl. Math. **3**, 348–355 (1946); these Rev. **7**, 300] in the analogous case of electromagnetic wave propagation.

C. J. Bouwkamp (Eindhoven).

Elasticity, Plasticity

***Novozhilov, V. V. Foundations of the nonlinear theory of elasticity.** Graylock Press, Rochester, N. Y., 1953. vi+233 pp. $4.00.

Translated from Osnovy nelineĭnoĭ teorii uprugosti [Gostehizdat, Moscow-Leningrad, 1948; these Rev. **12**, 651].

Csonka, P. Ein Lösungssystem der Grundgleichungen der Elastizitätstheorie. Acta Tech. Acad. Sci. Hungar. **2**, 487–490 (1952). (Russian summary)

An infinite series representation of the displacements of an isotropic, elastic body is given. The terms of the series contain sets of arbitrary constants as well as functions. The latter are termwise connected by recurrence relations. The terms can be split into translations and rotation. The author shows that if the series converge, the conditions placed on the functions are sufficient to yield a solution of the Navier equations. *G. H. Handelman* (Pittsburgh, Pa.).

Kušević, Rajko. Solution of the three-term equations of elasticity by the method of undetermined coefficients. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **276**, 83–99 (1949). (Serbo-Croatian)

Kušević, Rajko. Résolution des équations d'élasticité à trois termes par la méthode des coefficients indéterminés. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **5**, 25–31 (1952).

Abbreviated version of the paper listed above.

Jossa, Franco. Su una funzione delle deformazioni analoga a quella di Airy delle tensioni. Ricerca, Napoli **3**, no. 3–4 37–52 (1952).

The paper discusses the construction of a function $\Phi(x, y)$ [x and y being rectangular cartesian coordinates], analogous to the Airy stress function $F(x, y)$, whose second order partial differential coefficients directly determine the strains in a problem of plane strain. This is simply achieved for homogeneous isotropic material, and there is no essential difference between the use of Φ or of F. However, the author incorrectly envisages the application of his analysis to problems in which the material is not supposed homogeneous and isotropic; the error is due to the fact that the equations of equilibrium are not then satisfied.

H. G. Hopkins (Providence, R. I.).

Föppl, Ludwig. Ein Mittelwertsatz der ebenen Elastizitätstheorie. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1952**, 215–217 (1953).

The following mean-value theorem is announced: Let S_0 be the generalized plane-stress solution for a simply connected plate (boundary Γ) under arbitrary tractions on Γ. Let S_1 be the solution for the same plate with a traction-free hole (boundary γ) subject to the same loading on Γ. Then the mean value of the tangential stress along γ of S_1 is equal to the mean value of the first stress invariant of S_0 over γ in the limit as the ratio of the maximum diameter of γ to the minimum distance between γ and Γ tends to zero. The body-force field is assumed regular.

The special cases of the circular and elliptic hole in an infinite plate are cited as in agreement with the theorem. Photoelastic evidence is referred to in connection with holes of different special shapes. In support of the general theorem an intuitive argument is presented, involving an appeal to Saint-Venant's principle, and the assumption that $\int_\gamma f ds = 0$ over any closed contour lying in a simply connected region throughout which $\nabla^2 f = 0$. The object of the argument is to show that a self-equilibrated system of tractions applied to the boundary γ of a hole in an infinite plate gives rise to a stress field, the first invariant $\nabla^2\phi$ of which obeys $\int_\gamma \nabla^2\phi ds = 0$, ϕ being the corresponding Airy function. That this is not always true can be seen from the following example: With reference to polar coordinates (r, θ), let $\phi = \sin 2\theta$; let γ consist of $(r=4,\ 0 \leqq \theta \leqq \pi/2)$, $(r=2,\ \pi/2 \leqq \theta \leqq 2\pi)$, and of the straight-line segments joining these two circular arcs; here $\int_\gamma \nabla^2\phi ds = 1$. *E. Sternberg* (Chicago, Ill.).

Arf, Cahit. Sur l'identité d'un problème de frontière libre en élasticité avec un problème d'écoulement. Bull. Tech. Univ. Istanbul **4** (1951), no. 1, 1–4 (1952). (Turkish summary)

A short paper illustrating the analogy between free stream lines in two-dimensional flow of a fluid and free boundaries having a constant hoop stress in plane stress problems in elasticity. The analogy is illustrated by the flow through an

aperture, and the loading required on the rigid boundaries to give constant hoop stress along the corresponding free boundaries in the plane stress problem. *R. M. Morris.*

Favre, Henry, et Gilg, Bernhard. La plaque rectangulaire fléchie d'épaisseur linéairement variable. Z. Angew. Math. Physik **3**, 354–371 (1952).

The differential equation for the deflection is established and solved for a simply supported plate in two ways, in each of which the deflection is expressed as an infinite power series in the rate of change of thickness and the differential equation is written for each coefficient of the series. The first method follows Navier in using a double sine series solution. The second method follows Lévy to obtain a single series. Both methods are applied to the plate under linearly varying pressure and numerical values are obtained for deflections and moments. *D. C. Drucker.*

Müller, W. Zur Theorie der rechteckigen Fundamentplatten und Pilzdecken. Ing.-Arch. **20**, 278–290 (1952).

The deflection of plates on supporting columns and of footings on elastic foundations is obtained in a rapidly convergent single, instead of the usual poorly convergent double, series. Cases solved include one-panel and continuous slabs, either uniformly loaded or acted upon by concentrated loads. *D. C. Drucker* (Providence, R. I.).

Prokopov, V. K. Problem of restrained bending of a rectangular strip. Akad. Nauk SSSR. Inženernyĭ Sbornik **11**, 151–160 (1952). (Russian)

The classical theoretical solution of a cantilever beam possesses the following shortcoming: The boundary conditions at the fixed end are satisfied only for the center of the cross-section. This does not affect the correctness of the solution further along the beam, but the stress distribution in the most interesting cross-section cannot be obtained. P. F. Papkovich [C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 334–338 (1940); these Rev. **2**, 332] and A. I. Lourye [Akad. Nauk SSSR. Prikl. Mat. Meh. **6**, 151–168 (1942); these Rev. **5**, 138] applied a homogeneous solution of the elasticity equations and obtained a more accurate picture at the fixed end. They used series of complex functions depending on complex roots of a transcendental equation. Their method is not very practical for actual computations.

The author of this paper presents a more convenient solution. His biharmonic stress function consists of series of real functions whose coefficients depend on real and imaginary parts of roots of a transcendental equation. The function is very ingeniously constructed. The stress function indicates a two-dimensional problem, and this is what the author means by restrained bending. He derives the general formulas and then gives an example of a long cantilever beam (the width is small as compared with the length), a bending moment applied at the free end. The stresses at the fixed-end cross-section are found and tabulated.

T. Leser (Lexington, Ky.).

Šaišmelašvili, V. N. Approximate computation of a sufficiently sloping spherical shell with a given deformation of the contour. Soobščeniya Akad. Nauk Gruzin. SSR. **10**, 609–614 (1949). (Russian)

The author considers a shell as in the title (i.e., open shell of small curvatures) with rectangular contour when projected on a plane. All four vertices are fixed in ball shape supports which act as hinges. The convex side of the shell is upside. The two simultaneous differential equations determining the stress function and the normal deflections for this kind of a shell are given and referred to the author's previous works [cf. same Soobščeniya **10**, 397–403 (1949)]. The given deformations on the contour follow the sine law. The author uses the method of finite differences, dividing the middle surface of the shell into sixteen rectangles. He sets his boundary-value problem in finite difference form and solves it easily. He claims that the error is less than 6%. The general theory is followed by an example of a square spherical shell. All the stresses and displacements for the chosen points are tabulated, and the vertical deflections for three shells of different curvatures are graphically illustrated. The formulas found for the example were used to find the stresses in a concrete shell whose dimensions and elasticity constants are given. It was found that for the chosen displacement (which seemed small) the tensile stress in a vertex exceeds the strength of material limit.

T. Leser (Lexington, Ky.).

Oniašvili, O. D. Application of a variational method to problems of vibrations and stability of a sloping shell. Soobščeniya Akad. Nauk Gruzin. SSR. **10**, 601–608 (1949). (Russian)

The solution of the problem of local stability and of vibrations of a sufficiently sloping shell (a sloping shell is an open shell of small curvatures) can be obtained from the integration of two simultaneous differential equations which determine the stress function and the normal displacements. These equations were given by V. Z. Vlasov [Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1947**, 27–52; these Rev. **10**, 653]. In the case of harmonic vibrations the stress function which satisfies boundary conditions must also satisfy two variational equations. The Vlasov differential equations are difficult to solve in cases of variable stresses and curvatures. The above conditions do not affect the variational equations, whose solution can be reduced to a simple integration of preassigned functions. Such solution allows one to determine the natural frequency of vibrations of an initially stressed shell, a shell free from initial stresses, and also the critical load in the absence of body forces.

The author devotes this paper to the construction of the above-mentioned preassigned functions which would satisfy the boundary conditions. These functions are linear combinations of the fundamental functions used in the transverse vibrationsof beams. The combinations depend on the boundary conditions and seven different cases of supports on the edges are investigated. None of the functions determined in this way is actually substituted in the variational equations, the author explains only the subsequent procedure. *T. Leser* (Lexington, Ky.).

Žgenti, V. S. On the computation of a thin sloping shell having the form of a paraboloid of revolution. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 331–334 (1952). (Russian)

A "sloping shell" is an open shell of small curvature. The author solves the elasticity problem for a shell as in the title, starting from the equations given by Nazarov [same journal **13**, 547–550 (1949); these Rev. **11**, 486]. In solving the problem the author utilizes methods developed by I. N. Vekua [New methods for solving elliptic equations, OGIZ, Moscow-Leningrad, 1948; these Rev. **11**, 598]. Vekua's methods involve the application of complex variables and analytic functions. The solution is very general because boundaries, loads, and supports are arbitrary. *T. Leser.*

Föppl, Ludwig. Spannungen und Formänderungen von Ringschalen mit elliptischen Meridianschnitten. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1952**, 75–92 (1953).

The present paper is concerned with the problem of a thin toroidal shell of elliptical cross-section, acted upon by uniform normal wall pressure. The author refers to an earlier study of this problem [R. A. Clark, T. I. Gilroy, and E. Reissner, J. Appl. Mech. **19**, 37–48 (1952); these Rev. **14**, 429] and offers an alternate solution. The basic idea of his approach consists in separating the wall pressure p in the form p_1+p_2, where p_1 is resisted by membrane action and p_2 by bending action in a straight elliptical tube. The ratio p_1/p_2 is finally determined by requiring that the change of length of minor axis is the same for both states of stress. [Since in this solution incomplete account only is taken of the pertinent stress-strain relations of the theory of thin shells, the solution obtained can be no more than an approximate solution. A decision concerning the degree of approximation must wait until a numerical evaluation of the present approximate solution becomes available.]

E. Reissner (Cambridge, Mass.).

Parkus, H. Wärmespannungen in Rotationsschalen mit drehsymmetrischer Temperaturverteilung. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **160**, 1–13 (1951).

The equations in the theory of shells of rotational symmetry of H. Reissner [Müller-Breslau Festschrift, Leipzig, 1912, pp. 181–193] and E. Meissner [Phys. Z. **14**, 343–349 (1913)] are treated for the case of stresses due to temperature differences. Particular solutions for the spheroidal (Reissner), conical and cylindrical shell (Meissner) are given. With help of these solutions the boundary-value problems may be solved following the pattern of the method of J. Geckeler [Forschungsarbeiten Gebiete Ingenieurwesens no. **276** (1926); Handbuch der Physik, Bd VI, Springer, Berlin, 1928, p. 238]. In a footnote the author mentions a paper of J. Benischek [Österreich. Ing.-Arch. **5**, 117–129 (1951); these Rev. **13**, 90] where temperature stresses in a toroidal shell are treated. In the case of the conical shell particular solutions for temperature stresses were given already by G. Eichelberg [Forschungsarbeiten Gebiete Ingenieurwesens no. **263** (1923)]. *R. Gran Olsson*

İnan, Mustafa. Shear center of semi elliptical cross section. Bull. Tech. Univ. Istanbul **4** (1951), no. 1, 25–28 (1952). (Turkish summary)

Sternberg, E., and Sadowsky, M. A. On the axisymmetric problem of the theory of elasticity for an infinite region containing two spherical cavities. J. Appl. Mech. **19**, 19–27 (1952).

A solution to the problem of the title is given in series form, using spherical dipolar coordinates based upon the Boussinesq stress-function approach employed extensively by Neuber and the present authors. Stresses and displacements are written explicitly in terms of the Boussinesq functions for the general axisymmetric case. The two spheres are the same size in the problems treated numerically and are subjected to uniform hydrostatic tension at infinity. The problem is reduced to the solution of an infinite system of algebraic equations accomplished approximately by the usual truncation process. Interference between the two spheres is shown to be small when the clear distance between them is one diameter. *D. C. Drucker.*

Sternberg, E., and Rosenthal, F. The elastic sphere under concentrated loads. J. Appl. Mech. **19**, 413–421 (1952).

As in the previous papers by Sternberg and Sadowsky, the Boussinesq stress-function approach is employed [in particular, see the paper reviewed above]. The sphere is acted upon by two equal and opposite diametral loads. A careful study is made of the proper form of the singularity at each load and the danger and significance of pseudo-solutions is brought out. The correct solution is taken as the limit solution for a uniformly distributed normal surface traction as the area of application shrinks to zero and the resultant force approaches the concentrated load. It is the addition of this limit condition which constitutes the most valuable contribution of the paper. The correct singularity is not identical with that arising at a concentrated load acting normal to a plane boundary. A pseudo-solution may be looked at as the correct solution plus any solution for the limit of a self-equilibrating system of forces.

Numerical computations for the normal stress on the plane of symmetry are compared with the photoelastic results of Frecht and Guernsey [NACA Tech. Note no. **2822** (1952)]. *D. C. Drucker* (Providence, R. I.).

***Sternberg, E., Eubanks, R. A., and Sadowsky, M. A. On the axisymmetric problem of elasticity theory for a region bounded by two concentric spheres.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 209–215. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The available treatments of the boundary-value problems of elasticity theory for a spherical shell are somewhat limited in their usefulness as far as actual numerical evaluations are concerned. This paper contains an exact series solution in explicit form for the stresses and displacements in a spherical shell under arbitrary axisymmetric surface tractions. The solution is based upon the Boussinesq stress-function approach referred to spherical polar coordinates and covers also the limiting case of the solid sphere as well as that of a spherical cavity in a medium of infinite extent. In particular, the known closed solution corresponding to a traction-free cavity is re-established systematically. (Author's abstract.)

R. M. Morris (Cardiff).

Chong, Frederick. Indentation of a semi-infinite medium by an axially symmetric rigid punch. Iowa State Coll. J. Sci. **26**, 565–579 (1952).

The work of Harding and Sneddon employing the Hankel transform of Love's stress function is extended to the case where the frictionless rigid indenting surface is expressible in the form z=a fourth degree polynomial in r. Small deformations are assumed and τ_{rz} is taken as zero over the circular area of contact in the plane of the surface prior to indentation. Results are obtained in closed form. Numerical details are worked out for the spherical indenter.

D. C. Drucker (Providence, R. I.).

Ziemba, St. Full circular cylinder of a finite length, subjected to an axial compressive load. Arch. Méc. Appl., Gdansk **3**, 165–210 (1951). (Polish. English summary)

This work seems to be an unabridged dissertation for some academic degree. The preface gives all the basic equations of the theory of elasticity in rectangular and cylindrical coordinates, and also gives a list of harmonic and biharmonic functions. The second section considers the Boussinesq problem of semi-infinite space compressed by a concentrated load. The third section deals with a semi-infinite

cylinder compressed by a concentrated force, but the stress function is the same as for the semi-infinite space. The boundary conditions on the lateral face of the cylinder are not satisfied, which demonstrates probably that the stress function cannot be chosen arbitrarily. The remaining three sections deal with a finite compressed cylinder. The problem is solved three times, each time a different stress function is used. The first solution does not satisfy boundary conditions; the remaining two satisfy them. *T. Leser.*

*van Wijngaarden, A. **Ut tensio sic vis.** Anniversary Volume on Applied Mechanics dedicated to C. B. Biezeno, pp. 214–222. N. V. De Technische Uitgeverij H. Stam, Haarlem, 1953. (English)

The author sets up the equations for the pure bending of a thin rod which is helical in the unloaded and loaded states. He observes that for purely axial load a spring of initial slope zero and with free end able to rotate will extend without change of radius and in amount strictly proportional to the load if and only if $GI_t = EI_y$, where I_y and I_t are moments of inertia of the cross-section and G and E are the shear and Young's moduli of the material. For a spring with end constrained so as not to rotate, the radius necessarily changes and the extension cannot be strictly proportional to the load at large extension. For this case, however, the author suggests connecting two springs of the former type but opposite sense of rotation in series; the rotations then cancel, but the linearity of the load-extension curve is preserved. [The author makes no attempt to connect his work with the extensive literature on approximate theories of helical springs.] *C. Truesdell* (Bloomington, Ind.).

Ōkubo, H. **The torsion of spiral rods.** J. Appl. Mech. **20**, 273–278 (1953).

With the transformation $x+iy=(x'+iy')e^{ikz}$, where k is related to the inclination of the helix, the author transforms the equilibrium equation and stress compatibility relations to systems independent of the axial coordinate z. By neglecting certain small quantities of second order, the resulting system contains the plane problems of torsion and plane stress with their corresponding harmonic and biharmonic stress functions. In the case of helical springs, the author finds the equilibrium and compatibility equations are satisfied by two stress systems, one of which is the system for bodies of revolution with axial symmetry and the other the stress system for torsion of shafts with variable circular sections. As an illustration of the latter, the author uses Gohner's approximate solution for the ring shear stresses in a close circular section helix and obtains the remaining stresses τ_{rr}, $\tau_{\theta\theta}$, τ_{zz}, $\tau_{r\theta}$ by successive approximations. Maximum values of critical stresses are tabulated.

D. L. Holl (Ames, Iowa).

*Golomb, Michael, and Rosenberg, R. M. **Critical speeds of uniform shafts under axial torque.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 103–110. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The effect of axial torques on the critical speeds of uniform shafts with round cross section is investigated. The problem is solved in detail for shafts in self-aligning and in rigid-end bearings. It is found that critical speeds always decrease with axial torque. For a given applied torque, the effect of reducing the critical speed is larger for shafts in self-aligning bearings than for those in rigid bearings, and it is larger for lower modes than for higher ones. For the two cases considered in detail, numerical values of the first six critical speeds are given for torques ranging from zero to the lowest critical torque. In addition, formulas are given for determining the critical speeds with good accuracy, provided the modes and torques considered are not too high. A feature of the analysis is the determination of the roots of the eigenvalue equation by the use of symmetric functions.

D. L. Holl (Ames, Iowa).

Norzi, Livio. **Discussione intorno al principio variazionale per la instabilità elastica.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **11**, 187–192 (1951).

The author gives a brief discussion of his formulation of the variational principle of elastic stability. His results are similar to those of W. Prager [Quart. Appl. Math. **4**, 378–384 (1947); these Rev. **9**, 120] except that the energy term due to initial stresses contains only perturbed rotation terms. He claims that the other terms must be dropped to insure non-negativeness. [It is not clear to the reviewer that this requirement need be made.] A sketchy development of the proposed principle is given.

Rough comments on the use of the principle are made together with some statements concerning boundary conditions. The latter appear to be nothing other than the normal natural boundary conditions which arise in such problems. The method is applied to the buckling of an equi-flanged angle section under compression. *G. H. Handelman.*

Wittrick, W. H. **Buckling of oblique plates with clamped edges under uniform compression.** Aeronaut. Quart. **4**, 151–163 (1953).

An energy method is used to calculate the critical stress in the case of a clamped oblique plate (i.e., one having the shape of a parallelorgram) under uniform compression parallel to two of the sides. Numerical values for the buckling stress coefficient are obtained for plates having acute angles of 60° and 45° and side ratios varying between 3/5 and 5/3.

H. W. March (Madison, Wis.).

Zizicas, G. A. **Stability of thin elastic plates covering an arbitrary simply connected region and subject to any admissible boundary conditions.** J. Appl. Mech. **20**, 23–29 (1953).

This paper presents a new and completely general method of analysis for the determination of instability loads of a thin elastic plate with its middle surface an arbitrary simply-connected region; the plate is supported arbitrarily along its edge where, prior to instability, it is subjected to arbitrary applied forces acting in its middle surface. The method proceeds through the solution of the equation of neutral equilibrium in terms of a function of two independent complex variables, this being a procedure familiar elsewhere in the solution of certain two-dimensional problems of elasticity. In any particular case, appropriate characteristic values (which correspond to instability) of this equation are to be determined through the restrictions on the general solution necessary for the satisfaction of the boundary conditions (which depend upon the edge support), and it is proposed that these conditions be approximated by the method of least squares. The method is developed for orthotropic, as well as for isotropic, plates, and the force field in the plane of the plate, prior to instability, need not be homogeneous; moreover, a similar method may be devised for the discussion of the corresponding problem of free transverse vibrations. The method presented essentially stems from previous work of S. Bergman [see, for example, Math.

Ann. 86, 238–271 (1922)], and some previous results of Krzywoblocki [Quart. Appl. Math. 6, 31–52 (1948); 7, 236 (1949); these Rev. 10, 651] are recovered. Discussion of specific problems will be given elsewhere.

H. G. Hopkins (Providence, R. I.).

✶**Eringen, A. Cemal. Bending and buckling of rectangular sandwich plates.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 381–390. The American Society of Mechanical Engineers, New York, N. Y., 1952.

In treating problems dealing with sandwich plates certain simplifying assumptions have been made. Of these the most common are that all stresses in the core can be neglected except the shear stresses in planes perpendicular to the facings and that in certain cases the bending stiffness of the facings can be neglected. In this paper all six components of the stress tensor in the core are considered and the bending stiffness of the facings is taken into account. By the use of the theorem of the minimum total potential energy differential equations are obtained for the bendings of flat rectangular sandwich plates that have isotropic facings and cores and that are subjected to various types of loading and boundary conditions. The differential equations are solved for the case of a simply supported rectangular sandwich plate subjected to a combination of a transverse load and a compressive load on two opposite edges. It is found that the buckling load is higher than that predicted by previous theories. In certain cases it is considerably higher. A form of buckling that the author calls buckling with flattening of the core is found to occur in certain types of sandwich plates at low compressive loads. A simplifying assumption introduced in the analysis is that the displacements in both core and facings vary linearly with the distance from the midplane of the plate. Thus discontinuities in the components of the strain at the junctions of core and facings are neglected. According to an estimate made by the author the error thus introduced is small in the case of common types of sandwich. *H. W. March* (Madison, Wis.).

Wittrick, W. H., Myers, D. M., and Blunden, W. R. Stability of a bimetallic disk. Quart. J. Mech. Appl. Math. 6, 15–31 (1953).

The authors consider the problem how a bimetallic disk deforms when it undergoes a change of temperature. The general theory developed is applied to the particular case of a disk pressed into a spherical form, and it is shown that, in order to insure a snap action, the initial central rise of the disk must be greater than approximately twice the total thickness, a result which is independent of the radius of the disk and almost independent of the particular combination of the used metals. A formula is obtained for the mean temperature which the thermostat will maintain, being proportional to the initial central rise and the thickness and inversely proportional to the square of the radius and the difference between the coefficients of expansion. If the two metal sheets have approximately the same thickness, the mean temperature is nearly independent of their stiffness. The differential equations of the problem are essentially non-linear. In order to obtain the temperatures at which instability occurs, solutions were obtained with the aid of a differential analyser, the results enabling the amount of temperature hysteresis to be determined in any given case. In the general theory, the only assumptions made, apart from those involved in large deflection theory of plates, are that a temperature exists at which the disk is free from inertial stress and that the disk is axially symmetrical. In the particular application of the theory it is also assumed that Poisson's ratio of the two metals is the same. The authors point out that D. Yu. Panov has considered the same problem and derived differential equations of a type similar to those obtained in this paper [Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 603–610 (1947); these Rev. 9, 482]. The differences are due to the fact that Panov makes some additional simplifying assumptions which do not appear to be necessary. *R. Gran Olsson* (Trondheim).

Federhofer, K. Stabilität der Kreiszylinderschale mit veränderlicher Wandstärke. Österreich. Ing.-Arch. 6, 277–288 (1952).

Axial force is applied to a shell whose wall thickness, δ, varies continuously in the axial direction. Small deflection theory and axially symmetric buckling are assumed. A fourth-order differential equation in the variable $x\delta^3$ is obtained where x is the slope of the shell. An approximate form is also written for the case of slowly varying wall thickness. This equation is solved approximately by expanding the solution in powers of μ, the change in wall thickness divided by wall thickness at the thin end. The linear and quadratic variation in wall thickness are worked out. For small μ the critical load is said to be approximately $1+\mu$ times that for the uniformly thin shell. This answer seems reasonable only if the wave length for the buckled mode is the length of the shell. *D. C. Drucker* (Providence, R. I.).

Federhofer, Karl. Über die Eigenschwingungen der Kreiszylinderschale mit veränderlicher Wandstärke. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 161, 89–105 (1952).

The differential equations (classical small-strain theory) describing the vibrations of a cylindrical shell with variable wall thickness were presented by H. Egger [same S.-B. 147, 293–334 (1938), p. 297]. In the paper under review, these equations are simplified by the assumption that the wall thickness is a function only of the distance from one end of the tube (axial symmetry). A further simplification is achieved by the assumption that the thickness varies slowly. (Powers higher than the first of a thickness parameter are dropped.) The unperturbed problem (constant wall thickness) is first solved, the fundamental frequencies being tabulated for various values of certain dimensionless parameters. Then the perturbed problem (slowly varying wall thickness) is dealt with, a method being presented for finding the frequencies for pure twisting and for bending and twisting in terms of the solution to the unperturbed problem. Explicit solutions are given for linear and quadratic variations in the wall thickness. *H. D. Block.*

✶**Eringen, A. Cemal. On the non-linear vibration of circular membrane.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 139–145. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The author presents two non-linear partial differential equations governing the rotationally symmetric vibrations of a circular membrane, which equations are not restricted by the classical assumptions of small deformations and constant stresses. Neglecting all terms in the radial strain of higher order than the first but retaining the non-linearities in the radial slope, the author uses a perturbation procedure to solve the equations for the free vibration. An initial deflection of a zeroth order Bessel function type is as-

sumed, and two terms in the perturbation series are then obtained. The initial strain in the undeformed membrane, $\epsilon_0=(1-\nu)N_0/Eh$ where ν is Poisson's ratio, E the modulus of elasticity, N_0 the initial stress, and h the thickness of the membrane, is used as the perturbation parameter.

C. E. Langenhop (Ames, Iowa).

Maue, A.-W. Die Beugung elastischer Wellen an der Halbebene. Z. Angew. Math. Mech. **33**, 1–10 (1953). (English, French and Russian summaries)

This paper discusses the diffraction of harmonic waves in an isotropic elastic solid of infinite extent. These waves are normally incident on a half-planar crack at which they are reflected and diffracted. This problem is more complicated than the corresponding optical problem in that there are two boundary conditions to be satisfied on the crack. The author follows Clemmow's method [Proc. Roy. Soc. London. Sec. A. **205**, 286–308 (1951); these Rev. **12**, 884] in that he represents the scattered waves as an angular spectrum of plane waves. This leads to the formulation of the problem by means of a pair of dual integral equations. The solution of these integral equations is quite straightforward; it is equivalent to splitting an analytic function of a complex variable into a product of two factors, each containing a prescribed portion of the singular points. *E. T. Copson.*

Kellis-Borok, V. I. On the frequency equation of a multilayered elastic medium. Doklady Akad. Nauk SSSR (N.S.) **87**, 25–28 (1952). (Russian)

L'auteur considère un espace élastique composé de n couches planes, parallèles, homogènes, et isotrope+un demi-espace homogène et isotrope, les épaisseurs et autres caractéristiques de couches étant quelconques. Ce système est soumis à des tensions produisant des deformations, les unes et les autres étant fonctions continues des coordonnées. Sur la frontière du système les tensions normales sont nulles. En représentant les solutions des équations des ondes par les intégrales Fourier-Bessel dans le cas de symétrie axiale l'auteur forme l'équation caractéristique dont les solutions donnent les fréquencies des ondes. Il montre que les ondes libres sont possibles et que, à une distance suffisamment grande de la source, on peut avoir des ondes forcées d'un type spécial; leur vitesse de phase et l'intensité dépendent de la fréquence et de l'épaisseur des couches. La note donne les formules assez compliquées sans définitions et sans démonstrations. *V. A. Kostitzin* (Paris).

Pao, Yoh-Han, and Marin, Joseph. An analytical theory of the creep deformation of materials. J. Appl. Mech. **20**, 245–252 (1953).

The authors propose stress-strain relations for the creep deformation of an isotropic material under a fixed state of combined stress and at constant temperature. It is assumed that the instantaneous strain is the sum of three parts: the elastic strain that is related to the stress by Hooke's law, the transient creep strain and the "minimum rate creep strain". Between the transient creep strain, its time rate, and the stress, a relation is stipulated that lets the transient creep strain decrease exponentially under any constant state of stress. The time derivation of the minimum rate creep strain is supposed to depend on the stress in a non-linear manner. Because the material is isotropic and the stress remains constant during the considered deformation, the three parts of the strain have the same principal axes and these coincide with the principal axes of stress. Accordingly, the authors refer their stress-strain relations to those common principal axes; while the authors' fundamental assumptions would also apply to variable stresses, the resulting stress-strain relations would be much more complex.

W. Prager (Providence, R. I.).

von Mises, Hilda Geiringer. Fondamenti di una teoria matematica della plasticità. Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, 347–361 (1952)=Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **358** (1953).

This expository paper presents the fundamental equations of the general theory of perfectly plastic solids without making the usual assumption that the yield function and the plastic potential are identical. The equations are then specialized to the two-dimensional problems which are treated along the lines developed in an earlier paper [Z. Angew. Math. Mech. **32**, 379–387 (1952); these Rev. **14**, 702]. *W. Prager* (Providence, R. I.).

*Geiringer, Hilda. **On the general plane problem of a perfectly plastic body.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 539–545. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The paper is concerned with the general problem of plane plastic flow and the initial problem of plane plastic stress before the uniform thickness of the plate has changed appreciably. The results presented in the paper do not seem to go beyond those published in earlier papers by the author [Proc. Nat. Acad. Sci. U. S. A. **37**, 214–220 (1951); Quart. Appl. Math. **9**, 295–308 (1951); Z. Angew. Math. Mech. **32**, 379–387 (1952); these Rev. **13**, 186, 303; **14**, 702]; the basic equations are established with greater emphasis on the general three-dimensional problem, however.

W. Prager (Providence, R. I.).

*Drucker, D. C. **A more fundamental approach to plastic stress-strain relations.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 487–491. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The author defines work-hardening in terms of an external agency capable of doing a certain type of work. With the aid of this concept, it is shown that the loading surface is convex but may possess corners or pointed vertices at any point where the loading surface does not possess a unique normal vector. If a unique normal vector exists, then it is shown that the plastic strain increment vector must lie along this normal. This last result shows that the general Mises-Prager plastic potential stress-strain relations are valid. Similar results are obtained for the case of the ideal plastic body. (The reviewer does not follow the derivation of the basic equation [3].) *N. Coburn.*

Onat, E. T., and Drucker, D. C. Inelastic instability and incremental theories of plasticity. J. Aeronaut. Sci. **20**, 181–186 (1953).

Two theories have been proposed as a basis for the discussion of the instability of thin plate structures under load in the plastic range; these are described as "deformation" or "incremental" according as "total" or "incremental" strains are set out in the fundamental equations. Theoretical, and some direct experimental evidence, establish conclusively that the former theory is invalid; but, on the other hand, experimental values of the instability loads agree well with theoretical values based upon deformation theory, and these

disagree markedly with theoretical values based upon incremental theory. This apparent contradiction of the validity of incremental theory has been the subject of much controversy in recent years. The explanation of this contradiction has been thought to be connected mainly with the assumption that the structure is perfect in shape and perfectly loaded, this ideal state only being approximated, albeit perhaps quite closely, even under supposedly good test conditions. However, to date, such analyses as have been published were interpreted as definitely establishing that the necessary imperfections would have to be considerably larger than those actually present under good test conditions. The present paper, which confines attention to the qualitative solution of a special plastic instability problem seems to indicate that the failure of the previous attempts to explain the above contradiction in terms of initial imperfections may be due to errors in the approximation of fundamental stress-rate versus strain-rate equations. It is stressed that the instability of structures in the plastic range is, in general, not so simply treated as in the elastic range; this is due to the fact that even quite small initial imperfections or disturbances may, in some instances, be very important governing factors in the immediate behavior of the system. This paper should therefore stimulate the reinvestigation of the general problem in other important particular cases.

In detail, this paper studies the plastic instability load for a relatively long cruciform column under end compression, this problem being selected because here the above contradiction appears at its worst. For such a column, torsional failure is most likely, and this mode is studied. The authors treat the simplified problem of a thin shell of constant wall thickness and of cruciform shape, the normal cross-sectional shape being supposed maintained during twisting action; the stress-strain curve of the material is taken to be linear in the plastic range and, for simplicity, but with little pertinent restriction, the yield condition of von Mises is selected. Under certain conditions, implying a low rate of work hardening, it is found that extremely small initial imperfections of twist can account for the contradiction. Further, it is shown that once the yield limit is reached, no matter how small are the initial imperfections. no further increase in compressive load is required to continue the twisting action already set in; moreover, this process is unstable.

As the authors observe, consideration of the problem for an actual cruciform section made of a material with a more realistic mechanical behavior is a subject for future research. However, it may reasonably be conjectured that such further work, although important, would mainly result in analytical complications without affecting seriously the overall physical picture deduced from the study of the present simplified model. *H. G. Hopkins* (Providence, R. I.).

***Drucker, D. C., Greenberg, H. J., Lee, E. H., and Prager, W. On plastic-rigid solutions and limit design theorems for elastic-plastic bodies.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 533–538. The American Society of Mechanical Engineers, New York, N. Y., 1952.

Extremum theorems recently developed [Drucker, Greenberg and Prager; J. Appl. Mech. **18**, 371–378 (1951); Quart. Appl. Math. **9**, 381–389 (1952); these Rev. **14**, 431; **13**, 603] determine upper and lower bounds for the collapse (or limit) load of a supported elastic-plastic body subjected to external loads. Recent work [see, for example, R. Hill, The mathematical theory of plasticity, Oxford, 1950; these Rev. **12**, 303] on complete solutions of plane-strain boundary-value problems has determined some stress and velocity fields which satisfy the yield and equilibrium equations. If the above theorems are applied to such complete solutions the lower and upper bounds coincide, and so the true limit load for the elastic-plastic problem is obtained. Examples of such solutions given concern a pressure vessel (whose normal cross-section has a square outer boundary and a circular inner boundary) and a slab compressed between two, perfectly rough, rigid plates. In such problems it is virtually impossible to solve the complete elastic-plastic problem because of the complexity of the elastic-plastic boundaries. Some incorrect solutions, which satisfy all conditions in the plastic regions only, are also discussed. The authors remark that a study of the elastic-plastic solutions for small strains (e.g., those obtained numerically by relaxation methods), combined with a determination of the limit load by the present methods, may prove of value in obtaining a better understanding of the development of uncontained flow through contained plastic flow.

H. G. Hopkins (Providence, R. I.).

***Pell, W. H., and Prager, W. Limit design of plates.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 547–550. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The paper discusses the limit analysis of a thin plate, subjected to transverse load, supported at its edge, and made of a plastic-rigid material that obeys the Mises yield condition. The analysis is within the spirit of the simple bending theory of thin plates. Extremum theorems [see Drucker, Greenberg and Prager; J. Appl. Mech. **18**, 371–378 (1951); Quart. Appl. Math. **9**, 381–389 (1952); these Rev. **14**, 431; **13**, 603] yield lower and upper bounds to the magnitude of the transverse load P when plastic deformation of the plate first occurs. As an example, the problem of a simply-supported circular plate under uniformly distributed transverse load is studied in detail, a value for P, correct to within about 5%, being found.

The reviewer observes that considerable simplification may be expected in the analysis of the problem if the Tresca yield condition is used. In particular, the exact solution of the above special problem is then simply found [H. G. Hopkins and W. Prager, unpublished work].

H. G. Hopkins (Providence, R. I.).

Hopkins, H. G. The plastic instability of plates. Quart. Appl. Math. **11**, 185–200 (1953).

The author considers the plastic instability of rectangular plates under uniform normal edge stresses. It is assumed that the total rate of strain consists of two parts: (1) the elastic strain-rate which is related to the stress-rate by Hooke's law; (2) the plastic strain-rate which is related to the stress-rate by a loading function for which the Mises yield condition is valid. This leads to two expressions for the strain-rates depending upon whether the medium is in the loaded or unloaded condition. By using the same theory as is used in the thin-plate theory of elasticity, the author obtains two partial non-linear fourth-order differential equations for the stress-rate function, one equation applies when the material is loaded and the other is valid in the unloaded state. Application is made to two types of problems: (1) the Kármán problem in which the rate of change of the applied edge stresses at instability are zero; (2) the

Shanley problem in which the rate of change of the applied edge stresses at instability are due to plastic deformation.
N. Coburn (Ann Arbor, Mich.).

Pflüger, A. Zur plastischen Knickung gerader Stäbe. Ing.-Arch. **20**, 291–301 (1952).

This paper discusses the plastic instability of a uniform simply-supported beam, with doubly-symmetric cross-section, under end compression load. The analysis follows that due to F. R. Shanley [J. Aeronaut. Sci. **13**, 678 (1946); **14**, 261–268 (1947)], later extended by J. E. Duberg and T. W. Wilder [ibid. **17**, 323–327 (1950)] and by P. Cicala [ibid. **17**, 508–512 (1950)]. The author appears unaware of these latter two papers, and some of his results duplicate the results given there. *H. G. Hopkins.*

Inoue, Nobuo. Some cases of the axially symmetrical flow of perfectly plastic materials. J. Phys. Soc. Japan **7**, 512–518 (1952).

Axially symmetric problems in perfect plasticity are treated as statically determinate under the assumption that the principal stress acting on the meridian plane is equal to one of the principal stresses in the plane. The characteristic equations are obtained and applied to the problem of radial flow and flow in a conical plastic mass. *D. C. Drucker.*

Inoue, Nobuo. A new method of solution of the two-dimensional isostatical problem in the mathematical theory of plasticity. J. Phys. Soc. Japan **7**, 518–523 (1952).

The analogy between plastic flow and flow of a compressible fluid is discussed using the hodograph plane. The fluid has a Mach number of $2^{1/2}$ everywhere (the local velocity of sound in the fluid at rest is zero) and the pressure is proportional to the logarithm of its density. Equivalence of shocks and discontinuous solutions in plasticity is pointed out. In general, Mach lines correspond to slip lines, stream lines and potential lines to isostatics. Extension to three dimensions on the basis of the assumption in the paper reviewed above is also stated. *D. C. Drucker.*

Inoue, Nobuo. Application of gasdynamical method to soil mechanics and theory of plasticity. I. J. Phys. Soc. Japan **7**, 604–609 (1952).

The analogy is shown between non-viscous compressible gas in supersonic irrotational flow and the fully plastic plane problems for a soil or perfectly plastic mass without body force. In the former case $p=c\rho\gamma$, $-1<\gamma<1$; in the latter $p=c\log\rho$ [see the two preceding reviews]. The local Mach number is respectively $[2/(1-\gamma)]^{1/2}$ or $2^{1/2}$ everywhere.
D. C. Drucker (Providence, R. I.).

Inoue, Nobuo. Application of gasdynamical method to soil mechanics and theory of plasticity. II. J. Phys. Soc. Japan **7**, 610–618 (1952).

Statically determinate problems of plasticity are worked out in terms of the analogy discussed in the paper reviewed above. Examples given in detail are radial expansion and pressing between parallel and inclined plates which have been studied previously without use of the analogy.
D. C. Drucker (Providence, R. I.).

Shield, R. T. Mixed boundary value problems in soil mechanics. Quart. Appl. Math. **11**, 61–75 (1953).

The author considers a plastic material in plane strain which yields under the Coulomb yield condition. For this condition, the slip lines (called the failure lines) form fixed angles of magnitude less than $\pi/2$ with the direction of the algebraically greater principal stress. Since the slip lines lie in the characteristic directions (more properly, bicharacteristic directions) of the system consisting of the yield condition and the equilibrium relations, characteristic forms of these last relations may be obtained. These equations due to Kötter determine relations between the stress and the slope angle of the characteristics. However, the author's purpose is to apply the theory to the indentation of earth by a punch or a wedge. Here, zones of constant stress and radial shear as well as the velocity of indentation are assumed to be known. Thus, the forms of the characteristic relations in terms of the components of the velocity vector must be determined. This is done by assuming stress-rate of strain relations involving the yield condition as a plastic potential and then writing the characteristic relations in terms of the rate of change of the velocity vector. Cases when one family of slip lines are straight lines are discussed and applications are made to the punch and wedge identations of earth. *N. Coburn* (Ann Arbor, Mich.).

Jáky, J. Network of slip lines in soil stability. Acta Tech. Acad. Sci. Hungar. **6**, 25–39 (1953). (English, French, German and Russian summaries)

The article was arranged for publication by Á. Kézdi from the posthumous papers of Jáky. Its main emphasis is on the fact that simple solutions to soil pressure and stability problems do not satisfy all stress conditions in the presence of body forces. Velocity conditions are not considered and the problems are "statically determinate". Earth pressures are shown to cover a complete range of values between a maximum and minimum which are calculated for retaining walls. After detailed discussion, the conclusion is stated that computation of active earth pressure on the basis of a plane surface of slip continues to be a justified simplification. The reviewer believes that these points are made much clearer when the limit analysis point of view is adopted [cf. Drucker and Prager, Quart. Appl. Mech. **10**, 157–165 (1952); these Rev. **13**, 1007].
D. C. Drucker (Providence, R. I.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Glaser, Walter. Zur wellenmechanischen Theorie der elektronenoptischen Abbildung. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **159**, 297–360 (1950).

This is a comprehensive paper on diffraction theory of aberrations of electron optical system. Starting with Schrödinger's equation of an electron in an external electromagnetic field, the author has derived the characteristic (Hamilton's) equation of electron optics by means of Debye's transformation. In the second section the point characteristic is developed up to fourth order terms in the exit-pupil (aperture) coordinates x_a, y_a, by calculating first

the characteristic functions $E_{0a}(x_0, y_0; x_a, y_a)$ in the region where the electromagnetic field is present and $S_a(x_a, y_a; x, y)$ in the image region when the field vanishes. The wave amplitude function ψ of Schrödinger's equation in the image space is expressed with the aid of Kirchhoff's formula as follows:

$$(1)\quad \psi \simeq \left(\frac{kn}{2\pi h}\right)^{1/2} \int\int \exp\left[ikV(x_a, x_b; x, y)\right](k_0\Delta_0\Delta_1)^{1/2}dx_a dy_b$$

where $k=2\pi/\lambda$ (λ=de Broglie wave length), V is the phase function which is determined from E_{0a} and S_a,

$$\gamma_0\Delta_0=\partial(\alpha_0, \beta_0)/\partial(x_a, y_a), \quad \gamma_1\Delta_1=\partial(\alpha_1, \beta_1)/\partial(x_a, x_b),$$

and k_0 is proportional to the number of electrons within the conical bundle. In the fourth section the author evaluates (1) when the aberrations are very small. For an object on the x_0-axis and when (x, y) are referred to the Gaussian image point and by restricting V to linear and third order terms in x_a, y_a, the electron density in the image plane is given in terms of Lommel functions of first, second, and third order, the argument of which contains all the Seidel aberrations except spherical aberration and the two comas. Even if the latter three aberrations vanish, the wave function ψ still depends on the other five aberrations through the argument of the ~~Lommel~~ Bessel function of first order. The curves of equal intensity are in general ellipses about the Gaussian image point and reduce to circles when there are no aberrations of any kind. In section five V is expressed in terms of α, β, γ, by introducing the mixed characteristic. Finally, the author considers the effect of the aperture (exit-pupil) in the image side on the electron distribution in the image plane for an object on the axis. The electron density distribution when one neglects the spherical aberration term in (1) is given up to linear terms in B (spherical aberration constant) by:

$$(2)\quad I=|\psi|^2 \simeq \left(\frac{J_1(s)}{s}\right)^2 + \frac{2B\rho_a^2 J_1}{as}\left(J_2 - \frac{2J_1}{s}\right)$$

($s=kar\rho_a$, $x^2+y^2=r^2$, ρ_a=radius of exit-pupil, a=constant$=1/z_a$). The second term in (2) represents the contribution from the aperture and not from the deformation of the wave surface (front). [Reviewer's remark: The author's formula on p. 358 gives for the factor in parenthesis (J_2-J_1/s), which is incorrect.] *N. Chako.*

Grümm, Hans. Die Eigenladungs-Verbreiterung eines fokussierten Elektronenbündels. Ann. Physik (6) **11**, 131–144 (1952).

The geometrical treatment of an electron optical system predicts a constant intensity distribution on any plane in the image space which is parallel to the object plane. On the other hand, if the system is analized wave mechanically, the intensity is found to depend on the distance from the axis. For paraxial rays the current density distribution in the image space is expressed by two infinite series of ~~Lommel~~ Bessel functions the argument of which depends on the particular solutions of the paraxial ray equation. On the Fraunhofer plane the current density is given by the Airy formula and it is constant on the Gaussian plane. By considering the space charge effect, which is due to this non-uniform current density, the author obtains an integro-differential equation for the paraxial equation. On the assumption that the effect of space charge is small, he derives expressions for the displacement of the Fraunhofer and the Gaussian planes and for the change of magnification. The rest of the paper deals with the estimations of the upper limit of the space charge effect on the displacements, and numerical values are given for two different values of the accelerating potential.

N. Chako (Cambridge, Mass.).

Abelès, F. Sur la propagation des ondes dans les milieux stratifiés. Nuovo Cimento (9) **10**, supplemento, 214–223; discussion: 223–224 (1953).

Hufford, George A. A note on wave propagation through an inhomogeneous medium. J. Appl. Phys. **24**, 268–271 (1953).

The author is concerned with the electromagnetic field that results if a radio antenna is put near the earth, which is considered as a large nearly spherical body surrounded by an atmosphere whose refractive index is a function of both position and time. The usual methods are for very special cases. Here is sought a more direct and more intuitive approach by a modification of Kirchhoff's formula. Let φ be a scalar function satisfying the wave equation $\nabla^2\varphi+k^2\varphi=-4\pi\tau$, k being a function of position tending to k_0 at a large distance from the earth, τ being equal to zero at all but a single point P_0. At the surface S of the earth φ is to satisfy the boundary condition $\partial\varphi/\partial n=ik\delta\varphi$ and if R is the distance from the origin, for $R\to\infty$, $R\varphi$ is bounded and φ is to satisfy the Sommerfeld radiation condition $\lim_{R\to\infty} R(\partial\varphi/\partial R-ik_0\varphi)=0$. Let ψ be an arbitrary function, suitably regular and satisfying the following two conditions: (a) for P_1 very close to P_2, $\psi\sim 1/r_{12}$ (r_{12} is the distance P_1P_2); (b) for P_1 fixed and P_2 tending to infinity, $\lim_{R\to\infty} R(\partial\psi/\partial R-iK\psi)=0$, K being a constant. Using Green's theorem the author deduces the relation

$$\varphi=\int_{V_R}\tau\psi dv+\frac{1}{4\pi}\int_S \varphi\left(ik\delta\psi-\frac{\partial\psi}{\partial n}\right)da + \frac{1}{4\pi}\left\{\int_{V_R}\varphi(\nabla^2\psi+k^2\psi)dv+i(k_0-K)\int_{\Sigma_R}\varphi\psi da\right\},$$

V_R being the volume and Σ_R the surface of a sphere of radius R, and considers limits for $R\to\infty$. Now ψ is to be some trial solution of the problem; then we have here an integral equation for φ. The last term of the right-hand side, to be denoted by $T\varphi$, represents the effect of the inhomogeneity of the atmosphere. If we try to satisfy the equation $\nabla^2\psi+k^2\psi=0$ as closely as possible, $T\varphi$ is a measure of the degree to which we fail in this. The author gives another equation in which the surface integral over Σ_R is discarded:

$$T\varphi=\int_{V_R} A\nabla g dv+\frac{1}{4\pi}\int_V \varphi(\nabla^2\psi+k^2\psi-g\psi)dv.$$

Here the vector field $A(P_1, P_2)$ is a solution of

$$\nabla^2 A(P_1, P_2)=-\varphi(P_2)+\psi(P_1, P_2)$$

satisfying the condition at infinity

$$A=-[\varphi\psi/i(k_0+k)]r_1+O(R^{-1})$$

(r_1 is a unit radial vector) and $g(P_1P_2)$ is a suitably regular function, which for any fixed P_1 is zero for P_2 on S and which tends to $k_0^2-R^2$ as P_2 tends to infinity. If g makes $\nabla^2\psi+k^2\psi-g\psi$ nearly equal to zero, the second term in the last equation for $T\varphi$ is small compared with the first. The author gives some examples of trial solutions ψ, equivalent to well-known approximate methods. To proceed further we have to calculate the (small) value of $\nabla^2\psi+k^2\psi$, substi-

tute this value in the equation for $T\varphi$, and evaluate the resulting integral. This, however, seems as yet very difficult. *H. Bremekamp* (Delft).

Senior, T. B. A. The diffraction of a dipole field by a perfectly conducting half-plane. Quart. J. Mech. Appl. Math. 6, 101–114 (1953).

Using a method due to Bromwich the diffraction of a dipole field by a perfectly conducting half-plane may be expressed in terms of the known solutions of a scalar problem, but only for a dipole whose axis lies parallel to the diffracting edge. An alternative method is developed which is valid for all positions of the dipole and based on the resolution of the dipole field into plane waves whose diffraction by the sheet may be independently studied. The particular case considered is that of an electric dipole with its axis normal to the half-plane. *A. E. Heins.*

Sollfrey, William, and Shmoys, Jerry. Diffraction of electromagnetic waves by a plane wire grating. II. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-41**, i+11 pp. (1952).

Variational expressions for the scattering matrix of a wire grating obtained by J. Shmoys [New York Univ., Washington Square College, Math. Res. Group, Res. Rep. no. **EM-18** (1950); these Rev. **12**, 65], are evaluated for the case of circularly cylindrical wires. The single scattering approximation is used as a trial field. *A. E. Heins.*

Güttler, A. Die Miesche Theorie der Beugung durch dielektrische Kugeln mit absorbierendem Kern und ihre Bedeutung für Probleme der interstellaren Materie und des atmosphärischen Aerosols. Ann. Physik (6) **11**, 65-98 (1952).

The author obtains by standard methods the solution of the equation $\Delta u+m^2k^2u=0$ which behaves like a plane wave at infinity. Here k^2 is fixed and m^2 is a piecewise constant function in the three regions: $0<r<R_0$, $R_0<r<R$, $R<r$. *B. Friedman* (New York, N. Y.).

Magnus, Wilhelm. On the scattering effect of a rough plane surface. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-40**, i+15 pp. (1952).

A layer of dipoles is used to represent the roughness of a perfectly conducting plane surface. A random distribution of the dipoles is introduced by the assumption that the energy scattered by the dipoles shall show an additive behavior. The parameters of the problem are restricted by the assumption that the distance d between the source of radiation and the point of observation is large. The wavelength may be of the order of magnitude of the roughness, that is, of the height σ of the scattering dipoles above the plane. Let h and η be the distances from the reflecting plane of the sending dipole and of the point of observation. Then the scattered energy is computed as a function of d, h, η, σ for horizontal and for vertical polarization. The formulas involve an undetermined factor which depends only on the density of the dipole distribution and on the ratio of the amplitudes of the incoming wave and of the wave emitted by a single scattering dipole. *A. E. Heins.*

Ivanov, Yu. P. On the scattering of electromagnetic waves by an uneven surface. Doklady Akad. Nauk SSSR (N.S.) **87**, 719–722 (1952). (Russian)

The author considers the incidence of a plane wave on a doubly-corrugated surface, the lower medium being of finite conductivity. The resultant field is expressed as a double series of plane waves. The amplitudes of the latter are estimated in the special cases of sinusoidal and trochoidal cross-sections, significant differences being found between the two cases. The author follows the method and the simplifications used by L. M. Brehovskih [same Doklady (N.S.) **81**, 1023–1026 (1951); these Rev. **13**, 605] who, however, restricted himself to the case of a perfectly conducting surface. *F. V. Atkinson* (Ibadan).

Bazer, J., and Karp, S. N. Propagation of plane electromagnetic waves past a shoreline. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-46**, i+64 pp. (1952).

The problem of propagation of electromagnetic waves over a land-sea surface is treated. In particular, the authors consider the field which arises from the diffraction of a plane wave (or ground wave) whose direction of propagation is normal and whose magnetic vector is parallel to the shoreline of a land-sea surface. The sea is treated as a perfect conductor, and at the air-land interface the customary impedance boundary condition is imposed. Mathematically, the problem of determining these fields involves the solution of a mixed boundary value problem for the steady state wave equation. Integral representations of this solution are obtained essentially by means of the methods of Wiener and Hopf. Far field approximations are found in both the illuminated and shadow regions. Conditions are given which insure the uniqueness of the field. In addition, some comments are made regarding the possibility of coastal refraction in this formulation. *A. E. Heins.*

Weyl, Hermann. Die natürlichen Randwertaufgaben im Aussenraum für Strahlungsfelder beliebiger Dimension und beliebigen Ranges. Math. Z. **56**, 105–119 (1952).

The author extends his results [Math. Z. **55**, 187–198 (1952); these Rev. **14**, 225] on existence and uniqueness of solutions of the scalar wave equation, to m-dimensional skew-symmetric forms in an n-dimensional space. The proof includes Maxwell's equations in an exterior domain as a special case. The notation is that of the exterior derivative of E. Cartan, and the construction of the "capacity" matrix is simplified by use of a device of C. Müller [Math. Z. **56**, 80–83 (1952); these Rev. **14**, 518]. Since the construction for the existence is made to depend on a ground solution, the latter part of the paper is devoted to a proof that the order of singularity present in the kernel is one acceptable in the Fredholm theory. *W. K. Saunders.*

Aden, A. L. Back-scattering of electromagnetic waves from spheres and spherical shells. Air Force Cambridge Research Center, Cambridge, Mass. Geophysical Research Papers, no. **15**, 42 pp. (1952).

The classical theory of the scattering of a plane electromagnetic wave by a sphere is reviewed, and the difficulties in getting numerical answers from the formal solution are discussed. [There is no reference to the important work of H. C. van de Hulst, Optics of spherical particles, Dissertation, Utrecht Univ., 1946, nor to the monumental tables of Gumprecht and Sliepcevich, Tables of light-scattering functions for spherical particles, Engrg. Res. Inst., Univ. of Michigan, 1951; these Rev. **13**, 585.] The scattering amplitude coefficients occurring in the series solution are expressed in terms of the function

$$\delta_n(x) = (d/dx) \log [x^{1/2} Z_{n+1/2}(x)],$$

where $Z_{n+1/2}$ is a cylinder function of half-integer order. Numerical computation of $\delta_n(x)$ is effected by using the recurrence relation [cf. Infeld, Quart. Appl. Math. 5, 113–132 (1947); these Rev. 9, 125]

$$\delta_n(x)=\frac{x^2+nx\delta_{n-1}(x)-n^2}{nx-x^2\delta_{n-1}(x)}.$$

Numerical results for one special case of dielectric constant, $\xi_r=81-7.8j$, at $\lambda=16.230$ cm are displayed in a number of graphs for $0\leqq\alpha=2\pi a/\lambda\leqq 6$, where a is the sphere's radius. Comparison of theoretical and experimental results for the back-scattering from water spheres. Extension of theory to scattering of plane wave by sphere with concentric spherical shell. *C. J. Bouwkamp* (Eindhoven).

Power, G. Forces on the boundary of a dielectric. Pacific J. Math. 3, 233–246 (1953).

It is shown here how to find the forces caused by the refraction of the lines of electrostatic force at a surface of discontinuity separating media of different specific inductive capacities. The cases of a homogeneous circular dielectric cylinder and of a homogeneous dielectric sphere in a general field are discussed in detail. *E. T. Copson.*

van Bueren, H. G. On the attraction between a perfectly conducting plate and a thin perfectly conducting cylinder. Nederl. Akad. Wetensch. Proc. Ser. B. 55, 493–499 (1952).

To compute the interaction energy and corresponding attractive force between a wire and a perfectly conducting plate, the author evaluates the change of electromagnetic zero-point energy from the solution of the classical electromagnetic problem. He assumes a cubic cavity bounded by perfectly conducting plates and introduces a very thin perfectly conducting cylindrical wire. By a method of perturbation, the changes in resonant frequency of any possible mode due to the introduction of the wire is computed for general direction of incidence of the electromagnetic field. The change in zero-point energy is related to the sum total of these changes in resonant frequencies in accordance with quantum mechanical concepts. The interaction energy is then derived by substituting integration for summation, and from it the attractive force is determined if the wire is close to one boundary of the cavity. For a wire of radius 2μ at a distance of 5μ from the perfectly conducting plane the attractive force is of the same magnitude as the electrostatic force when a potential difference of about 0.0003 volts is applied between them. Thus, the force is too small to be measurable. *E. Weber* (Brooklyn, N. Y.).

Sexl, Theodor. Über die Äquivalenz von stationären Strömen mit magnetischen Doppelschichten. Acta Physica Austriaca 7, 198–199 (1953).

Rikitake, Tsuneji. Analyses of geomagnetic field by use of Hermite functions. Bull. Earthquake Res. Inst. Tokyo 30, 293–304 (1952). (Japanese summary)

Assuming that a magnetic anomaly observed on the earth's surface is caused by a horizontal magnetic layer of negligible thickness at depth D, the author considers the two-dimensional problem of interpretation postulating that the level curves of the anomaly $\Delta Z=f(x)$ are parallel straight lines. Under such restrictive assumptions the intensity of magnetization I of the plane layer is a function of x also: $I=F(x)$. The author expresses $F(x)$ in terms of coefficients K_m of the expansion of $f(x)$ in Hermite series. These coefficients are multiplied by definite integrals which do not depend on the anomaly and can be computed once and for all if the direction of magnetization I is known. Unfortunately, the integrals multiplying K_{2m} are essentially divergent. The author then replaces the first expansion by the expansion of the corresponding anomalous potential in Hermite series and uses the coefficients of that expansion to express $F(x)$. The expression of $F(x)$ contains again the unknown direction of I. *E. Kogbetliantz.*

Hines, C. O. Generalized magneto-hydrodynamic formulae. Proc. Cambridge Philos. Soc. 49, 299–307 (1953).

On the so-called magneto-ionic theory of a plasma the equation governing the velocity $\mathbf{V}_r$ with which a charge complex having a given charge to mass ratio (e_r/M_r) moves in a field of electric intensity $\mathbf{E}$ and magnetic induction $\mathbf{B}$ through an uncharged medium having a velocity $\mathbf{U}$ is

$$(1)\quad \frac{\partial \mathbf{V}_r}{\partial t}+(\mathbf{V}_r\cdot\operatorname{grad})\mathbf{V}_r=\frac{e_r}{M_r}(\mathbf{E}+\mathbf{V}_r\times\mathbf{B})+K_r(\mathbf{U}-\mathbf{V}_r)$$

where K_r is a suitably defined "collision frequency" which is generally assumed to be a constant. For infinitesimal periodic oscillations with frequency ω in the presence of a primary magnetic induction $\mathbf{B}_0$, (1) gives

$$(2)\quad i\omega\mathbf{V}_r=(e_r/M_r)(\mathbf{E}+\mathbf{V}_r\times\mathbf{B}_0)+K_r(\mathbf{U}-\mathbf{V}_r).$$

The solution of this equation can be written in the form

$$(3)\quad \mathbf{V}_r=\mathbf{U}+[1/M_rK_r']\mathbf{F}^*+[e_r/M_r^3(K_r'^2+G_r^2)]\mathbf{F}^*\times\mathbf{B}_0 +[e_r^2/M_r^3K_r'(K_r'^2+G^2)](\mathbf{F}^*\times\mathbf{B}_0)\times\mathbf{B}_0$$

where $\mathbf{F}^*=e_r(\mathbf{U}\times\mathbf{B}_0+E)-i\omega M_r\mathbf{U}$ and $K_r'=K_r+i\omega$ and G_r is the gyrofrequency $e_r\mathbf{B}_0/M_r$. With $\mathbf{V}_r$ determined by (3) the current density, $\mathbf{J}$, and the frictional reaction, $\mathbf{F}$, on the background material exerted by the charge in its motion are given by

$$(4)\quad \mathbf{J}=\sum N_re_r\mathbf{V}_r \quad\text{and}\quad \mathbf{F}=\sum N_rM_rK_r(\mathbf{V}_r-\mathbf{U})$$

where N_r denotes the concentration of the r-species of ions. In this paper the author seeks solutions of (3) and (4) compatible with Maxwell's equations which correspond to the propagation of a wave in the z-direction with the space time dependence $\exp i(\omega t+\beta z)$, and shows that

$$\beta^2-\omega^2\epsilon\mu=-\omega\mu\sum\frac{N_re_r^2}{M_r(\omega\pm G_r-iK_r)} +\frac{\omega\mu\sum[N_re_rK_r/(\omega\pm G_r-iK_r)]^2}{\omega\rho-i\sum N_rM_rK_r(\omega\pm G_r)/(\omega\pm G_r-iK_r)}$$

where ρ denotes the unperturbed mass density and ϵ and μ are the dielectric constant and the magnetic permeability, respectively. Various special cases of the foregoing solution are considered. *S. Chandrasekhar* (Williams Bay, Wis.).